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# Electric vector potential formulation in electrostatics: analytical treatment of the gaped surface electrode

**Robert Salazar**[1,2,a], **Camilo Bayona-Roa**[2,3,4], **Gabriel Téllez**[1]

[1] Departamento de Física, Universidad de los Andes, Bogotá, Colombia
[2] Universidad ECCI, Bogotá, Colombia
[3] Facultad de Ingeniería, Universidad Nacional de Colombia, Bogotá, Colombia
[4] Centro de Ingeniería Avanzada Investigación y Desarrollo, CIAID, Bogotá, Colombia

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**Abstract** The electric vector potential $\mathbf{\Theta}(\boldsymbol{r})$ is a legitimate—but rarely used—tool to calculate the steady electric field in charge-free regions. It is commonly preferred to employ the scalar electric potential $\Phi(\boldsymbol{r})$ rather than $\mathbf{\Theta}(\boldsymbol{r})$ in most of the electrostatic problems. However, the electric vector potential formulation can be a viable approach to study certain systems. One of them is the gaped surface electrode (SE): a planar finite region $\mathscr{A}_-$ kept at a fixed potential $V_o$ with a gap of thickness $\nu$ to the remaining grounded field. In this document, the Helmholtz decomposition theorem and the electric vector potential formulation are used to provide integral expressions for the surface charge density and the electric field of the gaped SE of arbitrary contour $\partial\mathscr{A}$. It is shown that the electric field of the gaped circular SE in the $\mathbb{R}^3$ space can be obtained from averaging the gapless SE solution over the gap. Even though the approach is illustrated with the circular SE, the strategy could be used in other geometries if the corresponding gapless solution is known. Analytic results are in agreement with numerical approximations of the electrostatic problem via Finite Element Method. Finally, the magnetic analogue of the gaped SE is provided.

## 1 Introduction

Calculation of electric field via the scalar electric potential $\Phi(\boldsymbol{r})$ is a standard approach in in electrostatics. However, the steady electric field in charge-free regions simplifies both to being an irrotational $\nabla \times \boldsymbol{E} = 0$ and divergence-free $\nabla \cdot \boldsymbol{E} = 0$ field. Hence, an electric vector potential can be associated with this field, in the sense of

$$\boldsymbol{E} = \nabla \times \mathbf{\Theta}(\boldsymbol{r}) \quad \text{if} \quad \rho(\boldsymbol{r}) = 0,$$

where $\rho(\boldsymbol{r})$ is the charge density. This implies that the electric vector potential satisfies the Laplace equation

$$\nabla^2 \mathbf{\Theta}(\boldsymbol{r}) = 0 \quad \text{if} \quad \rho(\boldsymbol{r}) = 0,$$

when a Coulomb-like gauge condition $\nabla \cdot \mathbf{\Theta}(\boldsymbol{r}) = 0$ is imposed. Even though using the electric vector potential as the electrostatic problem unknown is a valid strategy in some

[a] e-mail: rp.salazar84@uniandes.edu.co (corresponding author)

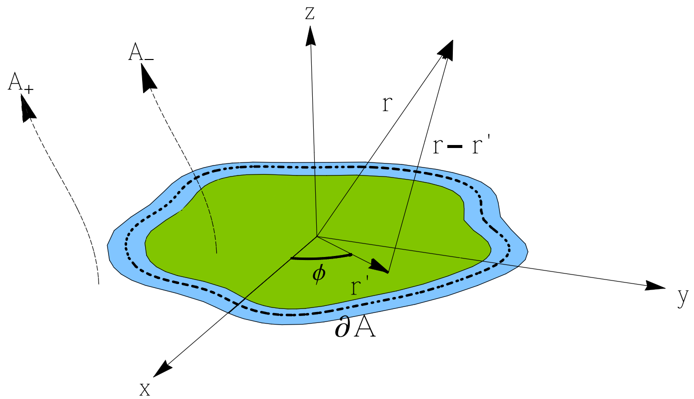


**Fig. 1** The gaped SE is composed by two flat sheets at different electric potentials separated by a gap in between

electrostatics settings, the scalar potential has been the preferred approach in the literature.[1] However, in the literature there are studies where the use of the electrical vector potential shows certain advantages, e.g., in the implementation finite element method for electrostatics [1] or in some engineering problems [2,3]. As we shall show in this document, the vector potential $\boldsymbol{\Theta}(\boldsymbol{r})$ is a good choice to describe the gaped SE: an infinite conductor sheet laying on the $xy$-plane and having a constant potential $V_o$ region $\mathscr{A}_-$ separated by a gap $\mathscr{G}$ from the grounded region $\mathscr{A}_+$ (see Fig. 1). Introducing a cylindrical system of coordinates over the SE plane, the gap can be defined as follows

$$\mathscr{G} = \{(r, \phi, 0) : \quad \mathscr{R}(\phi) - \nu/2 < r < \mathscr{R}(\phi) + \nu/2 \quad \forall \quad \phi \in [0, 2\pi)\},$$

with $(\mathscr{R}(\phi), \phi)$ and $(\mathscr{R}(\phi) \pm \nu/2, \phi)$ the parametric representations of contours $\partial A$ and $\partial A_\pm$, respectively.

This system plays an important role—for example—in the study of SE radio-frequency ion traps, which can be modelled as arrays of consecutive gapless SE. These ion-trap networks are promising candidates for manufacturing large-scale quantum processors [4–13]. Experimentally, the first surface electrode ion trap was built in 2005 [14], since then other devices have been proposed to increase the time of confinement of the trapped ions, e.g., the double-layer SE traps [15–17] and quasi double layer SE traps [18].

In any case, the knowledge of the electric field in the $\mathbb{R}^3$ space and the surface charge density distribution on the $xy$-plane is required to optimize the practical applications of surface electrodes. In the standard approach, the problem to be solved for scalar electrostatic potential consists in the Laplace's equation

$$\nabla^2\Phi(\boldsymbol{r}) = 0, \qquad \boldsymbol{r} \in \mathfrak{D} = \left\{\boldsymbol{r} \in \mathbb{R}^3 : z > 0\right\}, \tag{1}$$

subjected to the following Dirichlet and Neumann boundary conditions:

$$\Phi(\boldsymbol{r}) = V_o \qquad \text{if} \quad \boldsymbol{r} \in \mathscr{A}_- \subset \left\{\boldsymbol{r} \in \mathbb{R}^2 : z = 0\right\}, \tag{2}$$

$$\Phi(\boldsymbol{r}) = 0 \qquad \text{if} \quad \boldsymbol{r} \in \left\{\boldsymbol{r} \in \mathbb{R}^2 : z = 0\right\} \setminus \mathscr{A}_- \cup \mathscr{G}, \tag{3}$$

$$\frac{\partial\Phi(\boldsymbol{r})}{\partial z} = 0 \qquad \text{if} \quad \boldsymbol{r} \in \mathscr{G}. \tag{4}$$

[1] A similar choice has to be made for magnetostatics, for which the vector of magnetic potential is usually a better unknown than the scalar magnetic potential in current-free regions

Several analytic treatments using the standard approach have dealt with problem defined by Eqs. (1)–(4) in diverse configurations: rectangular strip electrodes held at constant potential [19], ring-shaped SE traps [20,21], and gapless SE with angular-dependent potential [22].

In this document, we propose an alternative approach using the vector electric potential. The Helmholtz decomposition in combination with the Green's theorem are used to calculate the electric field of a gaped SE system. The main idea is to find the vector field of electric potential in the SE and to recover the electric field by straightforward application of the curl operator on $\boldsymbol{\Theta}$. The determination of electric vector potential in the points not belonging to the SE can be used as a the departure step to find the charge density $\sigma$ distribution on the SE. In this document it is illustrated how to use $\boldsymbol{\Theta}$ to find $\sigma$ of a circular SE, demonstrating that this system is globally neutral. We present an alternative derivation of the result found in [23] for the gapless ($\nu = 0$) SE, without invoking any analogy with magnetostatics. Additionally, the electric vector potential approach can be used to determine magnetic analogue between the gapped SE in a quite intuitive way. The originality lies in the extension of the analytical treatment to the complete spatial domain, not only restricted to the surface, which has not been investigated before.

The remaining sections are devoted to the application of the Helmholtz decomposition to the electrostatic problem, the analytic treatment of the gapless SE, its extension to the circular gaped SE, the comparison against numerical solutions of the electrostatic field, the extension to the gapped SE problem with a generic contour definition and finally, the link of this problem with magnetostatics. Some conclusions are given at the end of this document.

## 2 Helmholtz decomposition theorem

The Helmholtz theorem [24] [25, p. 147] [26,27] states that a continuously differentiable vector field $\boldsymbol{P}(\boldsymbol{r})$ in a bounded region $\mathfrak{D} \subset \mathbb{R}^3$ can be decomposed into an irrotational (curl-free) field and a solenoidal (divergence-free) field, as,

$$\boldsymbol{P}(\boldsymbol{r}) = -\nabla\zeta(\boldsymbol{r}) + \nabla \times \boldsymbol{\Omega}(\boldsymbol{r}), \qquad \boldsymbol{r} \in \mathfrak{D}.$$

If the field $\boldsymbol{P}(\boldsymbol{r})$ decreases asymptotically ($r \to \infty$), then the irrotational and solenoidal fields can be written, respectively, as

$$\zeta(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\mathfrak{D}} \frac{\nabla' \cdot \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r} - \frac{1}{4\pi}\int_{\partial\mathfrak{D}} \frac{\hat{n}' \cdot \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}^2\boldsymbol{r},$$

and

$$\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\mathfrak{D}} \frac{\nabla' \times \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r} - \frac{1}{4\pi}\int_{\partial\mathfrak{D}} \frac{\hat{n}' \times \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}^2\boldsymbol{r},$$

where $\hat{n}'(\boldsymbol{r}')$ is the normal outward vector of $\partial\mathfrak{D}$.

The electric field $\boldsymbol{E}(\boldsymbol{r})$ can be thought to be the differentiable vector field $\boldsymbol{P}(\boldsymbol{r})$, such that the Helmholtz's decomposition is given by

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\zeta(\boldsymbol{r}) + \nabla \times \boldsymbol{\Omega}(\boldsymbol{r}), \qquad \boldsymbol{r} \in \mathfrak{D} = \left\{\boldsymbol{r} \in \mathbb{R}^3\right\}. \tag{5}$$

Since $\boldsymbol{E}(\boldsymbol{r})$ is also subjected to the Gauss's law $\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \rho(\boldsymbol{r})/\epsilon_o$, its irrotational part can be calculated from

$$\zeta(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o}\int_{\mathfrak{D}} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r} - \frac{1}{4\pi}\int_{\partial\mathfrak{D}} \frac{\hat{n}' \cdot \boldsymbol{E}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}^2\boldsymbol{r}. \tag{6}$$

For the SE setting, the volume integral is zero since there is no charge density in $\mathscr{D} \subset \mathbb{R}^3$. However, there exists a charge density different from zero on $\partial\mathscr{D}_{z=0} \subset \mathbb{R}^2$. One can calculate that distribution in terms of the Cartesian reference system as

$$\sigma(x, y) = 2\epsilon_o \lim_{z \to 0^+} E_z(x, y, z), \tag{7}$$

and therefore, the expression (6) simplifies greatly to

$$\zeta(\boldsymbol{r}) = \frac{1}{8\pi\epsilon_o} \int_{\mathbb{R}^2} \frac{\sigma(x', y')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}x'\mathrm{d}y'.$$

On the other hand, because the steady electric field is irrotational itself, the $\boldsymbol{\Omega}(\boldsymbol{r})$ field can be obtained from

$$\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\hat{z} \times \boldsymbol{E}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}x'\mathrm{d}y'. \tag{8}$$

Using the Cartesian system, the cross-product of (8) can be written as $\hat{z} \times \boldsymbol{E}(\boldsymbol{r}') = (\partial_y \Phi(\boldsymbol{r}), -\partial_x \Phi(\boldsymbol{r}), 0)$, and the components of $\boldsymbol{\Omega}(\boldsymbol{r}) = (\Omega_x(\boldsymbol{r}), \Omega_y(\boldsymbol{r}), 0)$ give, respectively,

$$\Omega_x(\boldsymbol{r}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \partial_y \Phi(x', y', 0) \mathrm{d}x'\mathrm{d}y', \quad \text{and}$$
$$\Omega_y(\boldsymbol{r}) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \partial_x \Phi(x', y', 0) \mathrm{d}x'\mathrm{d}y'.$$

At this point, let us introduce the integration by parts. Specifically, let us integrate by parts with respect to the $y'$-coordinate the integral expression of the $x$-component of $\boldsymbol{\Omega}(\boldsymbol{r})$, for which the more convenient integral form is given by:

$$\Omega_x(\boldsymbol{r}) = \frac{1}{4\pi} \int_{\mathbb{R}} \left[ \lim_{y' \to \infty} \frac{\Phi(x', y', 0)}{|\boldsymbol{r} - \boldsymbol{r}'|} - \lim_{y' \to -\infty} \frac{\Phi(x', y', 0)}{|\boldsymbol{r} - \boldsymbol{r}'|} \right] \mathrm{d}x'$$
$$- \frac{1}{4\pi} \int_{\mathbb{R}^2} \Phi(x', y', 0) \frac{y - y'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}y'\mathrm{d}x',$$

since

$$\frac{\partial}{\partial y'} \left( \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) = \frac{y - y'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}.$$

The boundary conditions of the gaped SE problem Eqs. (2)-(4), together with the supposition that $\Phi$ vanishes at $y \to \pm\infty$, simplifies the integrated-by-parts expression to:

$$\Omega_x(\boldsymbol{r}) = -\frac{V_o}{4\pi} \int_{\mathscr{A}_-} \frac{y - y'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}x'\mathrm{d}y' - \frac{1}{4\pi} \int_{\mathscr{G}} \Phi(x', y', 0) \frac{y - y'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}x'\mathrm{d}y',$$

for which the scalar electric potential inside the gap is unknown. Indeed, let us denote as $\Phi_{\mathscr{G}}(x', y')$ the scalar electric potential in $\mathscr{G}$. The $x-$component of the solenoidal field can be written as:

$$\Omega_x(\boldsymbol{r}) = -\frac{V_o}{4\pi} \int_{\mathscr{A}_-} \frac{\partial}{\partial y'} \left( \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \mathrm{d}x'\mathrm{d}y' - \frac{1}{4\pi} \int_{\mathscr{G}} \Phi_{\mathscr{G}}(x', y') \frac{y - y'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}x'\mathrm{d}y'.$$

A similar treatment can be used to obtain the integral expression of $\Omega_y(\boldsymbol{r})$ in terms of the electric potential inside the gap. The result of applying this procedures gives

$$\Omega_y(\boldsymbol{r}) = \frac{V_o}{4\pi} \int_{\mathscr{A}_-} \frac{\partial}{\partial x'} \left( \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \mathrm{d}x'\mathrm{d}y' + \frac{1}{4\pi} \int_{\mathscr{G}} \Phi_{\mathscr{G}}(x', y') \frac{x - x'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}x'\mathrm{d}y'.$$

The complete vector field $\boldsymbol{\Omega}(\boldsymbol{r}) = \Omega_x(\boldsymbol{r})\hat{x} + \Omega_y(\boldsymbol{r})\hat{y}$ can be written in the more convenient vector form

$$\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{V_o}{4\pi}\int_{\mathscr{A}_-} \boldsymbol{\xi}(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}^2\boldsymbol{r}' + \frac{1}{4\pi}\int_{\mathscr{G}} \Phi_{\mathscr{G}}(\boldsymbol{r}')\boldsymbol{\xi}(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}^2\boldsymbol{r}', \tag{9}$$

by introducing the vector function

$$\boldsymbol{\xi}(\boldsymbol{r}, \boldsymbol{r}') := \left\{-\hat{x}\frac{\partial}{\partial y'}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) + \hat{y}\frac{\partial}{\partial x'}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\right\},$$

comprising directional derivatives between the Cartesian and cylindrical systems.

Noting that the derivatives in the first term of the right-hand side (RHS) of (9) can be computed in the equivalent way

$$\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{V_o}{4\pi}\int_{\mathscr{A}_-}\left\{\frac{\partial}{\partial x'}\left(\frac{\hat{y}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) - \frac{\partial}{\partial y'}\left(\frac{\hat{x}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\right\}\mathrm{d}x'\mathrm{d}y' + \frac{1}{4\pi}\int_{\mathscr{G}} \Phi_{\mathscr{G}}(\boldsymbol{r}')\boldsymbol{\xi}(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}^2\boldsymbol{r}',$$

this equation simplifies by using the Green's theorem.[2] Indeed, the double integral over the plane region $\mathscr{A}_-$ can be computed from the integral around the bounding region $\partial\mathscr{A}_-$ (supposed to be oriented positively counterclockwise) in the sense of

$$\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{V_o}{4\pi}\oint_{\partial\mathscr{A}_-} \frac{\hat{x}}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}x' + \frac{\hat{y}}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}y' + \frac{1}{4\pi}\int_{\mathscr{G}} \Phi_{\mathscr{G}}(\boldsymbol{r}')\boldsymbol{\xi}(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}^2\boldsymbol{r}',$$

or equivalently, as

$$\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{1}{4\pi}\left[V_o\oint_{\partial\mathscr{A}_-} \frac{d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} + \int_{\mathscr{G}} \Phi_{\mathscr{G}}(\boldsymbol{r}')\boldsymbol{\xi}(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}^2\boldsymbol{r}'\right].$$

Remarkably, the $\boldsymbol{\Omega}(\boldsymbol{r})$ field depends exclusively on the shape of $\partial\mathscr{A}$ though the contour integral. Also, the $\zeta(\boldsymbol{r})$ scalar and $\boldsymbol{\Omega}(\boldsymbol{r})$ vector fields are directly related to the scalar electric potential $\Phi(\boldsymbol{r})$ and the vector electric potential $\boldsymbol{\Theta}(\boldsymbol{r})$, respectively. These are valid fields that represent the electric field in the charge-free region:

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\Phi(\boldsymbol{r}), \quad \text{and} \quad \boldsymbol{E}(\boldsymbol{r}) = \nabla\times\boldsymbol{\Theta}(\boldsymbol{r}).$$

Adding the previous expressions (and dividing by 2, if wished), the electric field can be computed from

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\left(\frac{\Phi(\boldsymbol{r})}{2}\right) + \nabla\times\left(\frac{\boldsymbol{\Theta}(\boldsymbol{r})}{2}\right). \tag{10}$$

[2] The Green's theorem states that

$$\int_{\mathscr{A}}\left[\frac{\partial}{\partial x}M(x,y) - \frac{\partial}{\partial y}L(x,y)\right]\mathrm{d}x\mathrm{d}y = \oint_{\partial\mathscr{A}} L(x,y)\mathrm{d}x + M(x,y)\mathrm{d}y$$

which is the relationship between the line integral around the positively oriented closed curve $\partial\mathscr{A}$ and the double integral over the plane region $\mathscr{A}$ bounded by $\partial\mathscr{A}$.