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Electrostatic field of angular-dependent surface electrodes

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Abstract We present an analytic strategy to find the electric field generated by surface electrode SE with angular-dependent potential. This system is a planar region $\mathscr{A}$ kept at a fixed but non-uniform electric potential $V(\phi)$ with an arbitrary angular dependence. We show that the generated electric field is due to the contribution of two fields: one that depends on the circulation on the contour of the planar region—in a Biot–Savart-Like (BSL) term—and another one that accounts for the angular variations of the potential in $\mathscr{A}$. This approach can be used to find exact solutions of the BSL electric field for circular or polygonal contours of the planar region with periodic distributions of the electric potential. Analytic results are validated with numerical computations and the Finite-Element Method.

1 Introduction

The computation of the electric field generated by a stationary charge density distribution is a standard problem of physics and engineering. Conceptually, the problem is simple: the electric potential due to a density charge distribution can be found by solving the Laplace's equation. However, obtaining an exact solution can be difficult depending on the charge density's distribution in space. In principle, it is possible to find analytic solutions to this problem in very different settings [1–5]. A particular case has to do with a fixed but not uniform potential distribution over a boundary surface. This type of distribution of charges on the boundary surfaces implies to solve a complex Dirichlet problem. At best, this system is integrable, as it occurs with some axially symmetric distribution of charges on a sphere or a disk [6,7]. In those cases, standard strategies of separation of variables can be used to find the electric potential. However, the computation of the electric potential for more complicated geometries often requires numerical approximations as the only possible approach.

In this article, we describe a technique to compute the electric field generated by a planar region with a non-uniform electric potential and its closure as grounded, as it is shown in Fig. 1. Using a cylindrical reference system, we define $\boldsymbol{r} = (r, \phi, z)$ to be any position inside the domain $\mathfrak{D} = \left\{\boldsymbol{r} \in \mathbb{R}^3 : z \geq 0\right\}$. A non-uniform electric potential $V = V(\phi)$ is defined inside the planar region $\mathscr{A}$ which is located at $z = 0$ and enclosed by the curve

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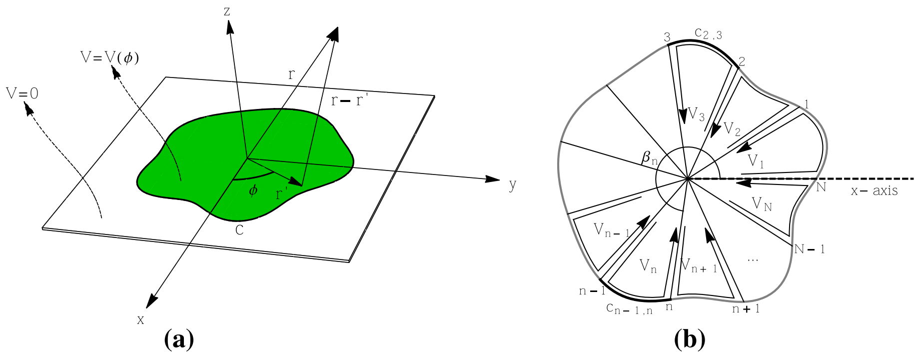

Fig. 1 **a** Planar region $\mathscr{A}$ of arbitrary contour c with a ϕ-dependent electric potential $V(\phi)$. **b** Discrete distribution of the potential on the sheet

c. The system in Fig. 1 is commonly referred as the *Gapless Plane Electrode* (GPE), and it plays an important role in the study of surface-electrode (SE) radio-frequency ion traps, since those can be modelled as an infinite plane gaplessly covered by an array of electrodes of arbitrary shape. Indeed, SE ion traps are a promising approach to build ion-trap networks suitable for large-scale quantum processing [8–15], and there has been an increasing interest to determine the analytic electric field of these structures. An analytical exact solution of the electric potential of a rectangular strip electrode held at constant voltage is presented in [16]. Ring-shaped SE traps have also been studied analytically for constant voltages in [17,18]. In general, the computation of the electric field in those systems can be achieved by numerical approaches. However, if the electric potential in $\mathscr{A}$ is constant, say $V = V_o$, then the electric field can be computed from:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{V_o}{2\pi} \oint_c \frac{\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}, \tag{1}$$

since the electric field at $z > 0$ is proportional to the magnetic field that would be observed if a current runs along c [19]. Hence, the Biot–Savart-like integral in Eq. (1) can be employed to drastically simplify the problem as it is shown in [17] both for a strip electrode and a ring-shaped SE trap.

If the potential in $\mathscr{A}$ is not constant, then Eq. (1) cannot be applied. Finding analytic solutions for this system is still a challenging problem. Most approaches rely on applying numerical methods to approximate the solution of the Laplace's equation [20–23].

The main objective of this work is to provide a generalization of Eq. (1) for angular-dependent potentials $V = V(\phi)$ on $\mathscr{A}$ which enable to find analytic solutions of the electric field. We focus our study on two specific settings where it can be applied: a circular region and a general non-intersecting polygonal region.

The remaining parts of this document are organized as follows: in Sect. 2, we derive an analytic expression to generate the electric field $\boldsymbol{E}(\boldsymbol{r})$ using an analogy between this electrostatic problem and magnetostatics. In that section, we shall demonstrate that the expression for the electric field can be written as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{1}{2\pi} \oint_c V(\phi') \frac{(\boldsymbol{r} - \boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} + \frac{1}{2\pi} \int_0^{2\pi} V(\beta) \partial_\phi \boldsymbol{f}(\beta, \boldsymbol{r}) \mathrm{d}\beta \quad \text{for} \quad z > 0, \tag{2}$$

where the line integral at the right side of the previous expression is analogous to the problem of computing the magnetic field along a closed loop c.[1] Here, the magnetostatic analogue corresponds to counter-intuitive situation where the loop carries a non-uniform electric current. [2] The second term in the right of Eq. (2) takes into account the electric field contributions due to the variations of $V(\phi)$ inside the region, where $\boldsymbol{f}$ is a vector field that depends on the shape of c. We stress out the fact that Eq. (2) *is valid for any closed loop c independently of its shape*, but the exact solution of this expression for arbitrary contours can be difficult. This is especially the case of the Biot–Savart-like term, which depending on the contour can be very hard to be solved. Hence, in Sect. 3, we perform a exact analytic expansion in the case of circular contours with any arbitrary but periodic potential functions. The application of this technique on SE settings having polygonal boundaries is demonstrated next in Sect. 4. We devote considerable effort in Sect. 5 to perform systematic comparisons of the analytic results against the numerical ones. Finally, conclusions are stated at the end of the document.

2 Electric field via Biot–Savart

In this section, we address the general expression to obtain the electric field inside the problem setting. We begin this section by recalling the basic definitions of the electrostatic problem: in that case, a steady electric field $\boldsymbol{E}$ has associated an electric potential $\Phi(\boldsymbol{r})$, such that $\boldsymbol{E} = -\nabla\Phi(\boldsymbol{r})$. Including the previous definition in the Gauss's law leads to the Poisson's problem:

$$\nabla^2\Phi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon_o} \quad \text{for} \quad \boldsymbol{r} \in \mathfrak{D},$$

where $\rho(\boldsymbol{r})$ is the volume charge density [6,7,25]. In this problem, $\rho(\boldsymbol{r}) = 0$ for $z > 0$, such that the Laplace's equation $\nabla^2\Phi(\boldsymbol{r}) = 0$, $\boldsymbol{r} \in \mathfrak{D}$ subjected to the boundary conditions:

$$\Phi(\boldsymbol{r}) = V(\phi) \quad \textbf{if} \quad \boldsymbol{r} \in \mathscr{A} \subset \left\{\boldsymbol{r} \in \mathbb{R}^2 : z = 0\right\}, \quad \text{and} \quad \Phi(\boldsymbol{r}) = 0 \quad \textbf{if} \quad \boldsymbol{r} \in \left\{\boldsymbol{r} \in \mathbb{R}^2 : z = 0\right\} \backslash \mathscr{A},$$

has to be solved in the domain. Naturally, the electric potential can be obtained from the solution of the Laplace's equation using Green's functions $G(\boldsymbol{r}, \boldsymbol{r}')$ [6]. The general solution for the Poisson's problem can written as:

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o}\int_{\mathfrak{D}} \rho(\boldsymbol{r})G(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}' + \frac{1}{4\pi}\oint_S \left[G(\boldsymbol{r}, \boldsymbol{r}')\frac{\partial\Phi}{\partial n'} - \Phi(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'}\right]\mathrm{d}^2\boldsymbol{r}'.$$

Here, the first term at the right hand vanishes, since $\rho(\boldsymbol{r}) = 0$ for $r \in \mathfrak{D}$. As usual, we may demand that $G(\boldsymbol{r}, \boldsymbol{r}')\big|_{z'=0} = 0$, by choosing the Green's function for the half-space $z > 0$ and considering the potential as a punctual charge at (x', y', z'), with $z' > 0$, plus the potential of an image charge placed in a symmetric position in the lower half plane, at

[1] Being only a an analogy between two different physical systems: the electrostatic where charges are static and the magnetostatic where current is time-independent.

[2] Generally speaking, an electric current along a wire is considered as constant. However, there are devices that *emulate* non-uniform distributed currents along wires (see for instance the setup in [24] which can be used in the experimental evaluation of pipe-lines' cathodic protection, the DC version of this setup would also emulate a steady non-uniform current along a wire). Another case concerns the electric current of a point-like charge moving in a loop: the electric current associated with this charge is not uniform.

$(x', y', -z')$. The Green's function can be stated as:

$$G(\boldsymbol{r}, \boldsymbol{r}') = \sum_{\sigma\in\{+1,-1\}} \frac{\sigma}{\sqrt{(x-x')^2 + (y-y')^2 + (z-\sigma z')^2}}.$$

Hence, the solution to the electrostatic problem reduces to evaluate the following integral:

$$\Phi(\boldsymbol{r}) = -\frac{1}{4\pi}\int_{\mathscr{A}} V(\phi')\frac{\partial G(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'}\mathrm{d}^2\boldsymbol{r}', \tag{3}$$

where $\hat{n}' = -\hat{z}$ is the outward normal of $\partial\mathfrak{D}$ on the $(z = 0)$-plane, and

$$\left.\frac{\partial G(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'}\right|_{z=0} = -\left.\frac{2(z-z')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right|_{z'=0}.$$

Solving the previous expression given the electric potential is not easy and usually numerical integration is needed. However, if the function $V(\phi)$ is constant, say V_o, then, from Eq. (3), the electric potential $\Phi_{\text{unif.}}(\boldsymbol{r})$ simplifies to:

$$\Phi_{\text{unif.}}(\boldsymbol{r}) = \frac{V_o}{2\pi}\int_{\mathscr{A}}\frac{(\boldsymbol{r}-\boldsymbol{r}')\cdot\hat{z}}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}^2\boldsymbol{r}' = \frac{V_o}{2\pi}\Omega(\boldsymbol{r}).$$

Authors of Ref. [19] remarked that the physical quantity $\Omega(\boldsymbol{r})$ is proportional to the *scalar potential* of a steady magnetic field $\boldsymbol{B}(\boldsymbol{r})$ [26,27] generated by the electric current i_o that flows along a closed arbitrary loop c. In that problem, the magnetic field could be computed from:

$$\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_o i_o}{4\pi}\nabla\Omega(\boldsymbol{r}) = \frac{\mu_o i_o}{4\pi}\oint_c\frac{\mathrm{d}\boldsymbol{r}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.$$

Similarly to the previous expression, one can compute the electric field from the definition of the scalar potential as:

$$\boldsymbol{E}_{\text{unif.}}(\boldsymbol{r}) = -\nabla\Phi_{\text{unif.}}(\boldsymbol{r}) = -\frac{V_o}{2\pi}\nabla\Omega(\boldsymbol{r}) = \frac{V_o}{2\pi}\oint_c\frac{\mathrm{d}\boldsymbol{r}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \quad \text{for} \quad z>0, \tag{4}$$

and therefore, the electric field can be computed by evaluating a Biot–Savart-Like (BSL) integral. Formally, Eq. (4) is only valid when V is kept as a constant in the region $\mathscr{A}$. The challenging work is to generalize this result for any potential $V(\phi)$ with an arbitrary angular dependence. To this aim, let us first consider the problem where $V(\phi)$ is not uniform, but varies discretely as follows: $V(\phi) = V_n \quad \textbf{if} \quad \phi\in(\beta_{n-1}, \beta_n)$, where $0<\beta_1<\beta_2<\cdots<\beta_N = 2\pi$ defines N consecutive sectors $\{\mathscr{A}\}_{n=1,\ldots,N}$ of uniform electric potential $\{V_n\}_{n=1,\ldots,N}$, where $\mathscr{A} = \mathscr{A}_1\cup\mathscr{A}_2\cdots\cup\mathscr{A}_N$. The boundary of each sector A_n is the loop $c_n\cup c_{n-1,n}\cup\tilde{c}_{n-1}$. Let us introduce an angular function $\mathscr{R}(\phi)$ that returns the planar region contour's radius and helps to define the position of the external curve in polar coordinates $c_{n-1,n} := \{(\mathscr{R}(\phi), \phi) : \beta_{n-1}<\phi\le\beta_n\}$. Therefore, the complete closed curve c can be defined as $c = \{(\mathscr{R}(\phi), \phi) : 0<\phi\le 2\pi\} = c_{1,2}\cup c_{2,3}\ldots\cup c_{N-1,N}\cup c_{N,1}$. The path c_n is a straight line from the point $(\mathscr{R}(\beta_n), \beta_n)$ to the origin and $\tilde{c}_n$ is its reversed trajectory. Since the potential of each sector is constant, we may use the superposition principle together with Eq. (4) to compute the electric field in the domain as:

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{n=1}^{N}\boldsymbol{E}_n(\boldsymbol{r}) = \frac{1}{2\pi}\sum_{n=1}^{N}\int_{c_n\cup c_{n-1,n}\cup\tilde{c}_{n-1}} V_n\frac{\mathrm{d}\boldsymbol{r}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.$$

We have denoted $\boldsymbol{E}_n(\boldsymbol{r})$ as the electric potential due to the nth sector of $\mathscr{A}$. It is even possible to split the integral of the previous equation into three different contributing terms:

$$\int_{c_n \cup c_{n-1,n} \cup \tilde{c}_{n-1}} V_n \frac{\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \int_{c_{n-1,n}} V_n \frac{\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \int_{c_n} V_n \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} - \int_{c_{n-1}} V_n \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.$$

Thus, the expression for the electric field expression can take the form:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{2\pi} \int_{\bigcup_{n=1}^{N} c_{n-1,n}} V(\phi') \frac{\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \frac{1}{2\pi} \sum_{n=1}^{N} V_n \left[\int_{c_n} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} - \int_{c_{n-1}} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \right]. \tag{5}$$

The first term on the right of Eq. (5) is the circulation over c, since $c = \bigcup_{n=1}^{N} c_{n-1,n}$. On the other hand, the sum of the straight lines' integrals can be written in a most convenient form by noting that:

$$\sum_{n=1}^{N} V_n \int_{c_{n-1}} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -V_1 \int_{c_0} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \sum_{n=1}^{N-1} V_{n+1} \int_{c_n} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.$$

Therefore:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) = {} & \frac{1}{2\pi} \oint_c V(\phi') \frac{\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \\ & + \frac{1}{2\pi} \left[-V_1 \int_{c_0} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \sum_{n=1}^{N-1} (V_n - V_{n+1}) \int_{c_n} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \right. \\ & \left. + V_N \int_{c_N} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \right], \end{aligned}$$

and using the periodicity conditions:

$$c_m = c_{N+m}, \quad \text{and} \quad V_{N+m} = V_m, \quad \forall \quad m \in \mathbb{Z}^0; \tag{6}$$

then, the electric field can be written as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{1}{2\pi} \oint_c V(\phi') \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \frac{1}{2\pi} \sum_{n=1}^{N} (V_n - V_{n+1}) \boldsymbol{f}(\beta_n, \boldsymbol{r}). \tag{7}$$

Here, we have defined the vector field:

$$\boldsymbol{f}(\phi', \boldsymbol{r}) := \int_0^{\mathscr{R}(\phi')} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times \mathrm{d}\rho' \hat{\rho}(\phi')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{8}$$

in Cartesian coordinates with $\hat{\rho}(\phi') = (\cos(\phi'), \sin(\phi'), 0)$. This term is a vector with dimensions of inverse length, which is analogous to the *magnetic field* generated by a unitary current