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Expansion Formula For the Magnetic Field of a Periodically Deformed Circular Current Loop

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Abstract

A method is derived to obtain an expansion formula for the magnetic field $\boldsymbol{B}(\boldsymbol{r})$ generated by a closed planar wire carrying a steady electric current. The parametric equation of the loop is $\mathscr{R}(\phi) = R + Hf(\phi)$, with R the radius of the circle, $H \in [0, R)$ the radial deformation amplitude, and $f(\phi) \in [-1, 1]$ a periodic function. The method is based on the replacement of the $1/|\boldsymbol{r}-\boldsymbol{r}'|^3$ factor by an infinite series in terms of Gegenbauer polynomials, as well as the use of the Taylor series. This approach makes it feasible to write $\boldsymbol{B}(\boldsymbol{r})$ as the circular loop magnetic field contribution plus a sum of powers of H/R. Analytic formulas for the magnetic field are obtained from truncated finite expansions outside the neighborhood of the wire. These showed to be computationally less expensive than numerical integration in regions where the dipole approximation is not enough to describe the field properly. Illustrative examples of the magnetic field due to circular wires deformed harmonically are developed in the article, obtaining exact expansion coefficients and accurate descriptions. The method can be also applied when the deformation function $f(\phi)$ is a linear combination of harmonic functions. In that case, we provide the exact formulas for the expansion coefficients of $\boldsymbol{B}(\boldsymbol{r})$. Error estimates are calculated to identify the regions in $\mathbb{R}^3$ where the analytical expansions perform well. Finally, first-order deformation formulas for the magnetic field are studied for generic even deformation functions $f(\phi)$.

Keywords: Magnetic Field, Gegenbauer Polynomials, Arbitrary current loop, Biot-Savart law.

1 Introduction

In 1820, the French physicists Jean Baptiste Biot and Felix Savart derived an expression for the magnetic field due to an electrical current i in a wire. This expression has become fundamental in magnetostatics and it is now commonly known as the *Biot-Savart law* [1,2]. It is

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_o i}{4\pi} \oint_{\mathsf{C}} \frac{d\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \tag{1}$$

where $\boldsymbol{r}$ is the observation point, $\boldsymbol{r}'$ is the position of the wire differential segment, μ_o is the permitivity of the medium, and $\mathtt{C}$ is the path of the wire. The depiction of this system is shown in Fig. 1. The law is only valid for stationary currents, and it plays a similar role to that of Coulombs' law in electrostatics.

Although the Biot-Savart law is commonly used in magnetostatic, other physical problems require solving integral expressions analogous to Eq. (1). One example is the case of vortex filaments in fluid mechanics [3–9], where the flow vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}(\boldsymbol{r})$ is concentrated along a filament C, with $\boldsymbol{v}(\boldsymbol{r})$ the velocity field. For incompressible fluids satisfying $\nabla \cdot \boldsymbol{v}(\boldsymbol{r}) = 0$, the vorticity expression leads to the Poisson's problem $\nabla^2 \boldsymbol{v}(\boldsymbol{r}) = -\nabla \times \boldsymbol{\omega}(\boldsymbol{r})$[1].

Vortex filaments	**Magnetostatics**
Velocity $\boldsymbol{v}(\boldsymbol{r})$	Magnetic field $\boldsymbol{B}(\boldsymbol{r})/\mu_o$
Vorticity $\boldsymbol{\omega}(\boldsymbol{r}) = \nabla \times \boldsymbol{v}(\boldsymbol{r})$	Ampere's law (Density current) $\boldsymbol{J}(\boldsymbol{r}) = \nabla \times (\boldsymbol{B}(\boldsymbol{r})/\mu_o)$
Continuity $\nabla \cdot \boldsymbol{v}(\boldsymbol{r}) = 0$	Gauss's Law $\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0$
Circulation $\Gamma = \int_S \boldsymbol{\omega}(\boldsymbol{r}) \cdot d\boldsymbol{S}$	Electric current $i = \int_S \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{S}$

Table 1: Physical analogy between magnetostatics and vortex filaments in incompressible fluid flows.

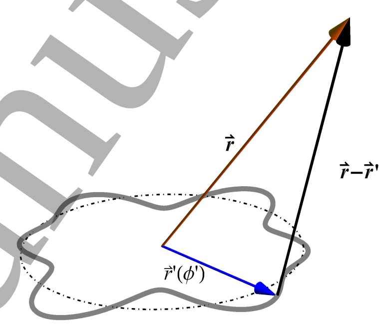

Fig. 1: Generic loop.

Now, if the vorticity field is regular enough, such that its derivatives are bounded and unique in $\mathbb{R}^3$, then the solution of the previous Poisson's equation is the divergence-free velocity field

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{K}(\boldsymbol{r}-\boldsymbol{r}') \times \boldsymbol{\omega}(\boldsymbol{r}') d\boldsymbol{r}' = \frac{\Gamma}{4\pi}\oint_{\mathtt{C}} \frac{(\boldsymbol{r}-\boldsymbol{r}') \times d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \qquad \text{with} \qquad \boldsymbol{K}(\boldsymbol{r}) = -\frac{\boldsymbol{r}}{4\pi|\boldsymbol{r}|^3},$$

and Γ the strength of the vortex. Table 1 lists the physical analogy between magnetostatics and the vortex filaments in fluid flows.

Even when the Biot-Savart law describes magnetostatic fields, it also plays and important role in the design and theory of wire antennas and radiation problems [10–13]. Another mathematically-equivalent physical system is the Gapless Surface Electrode (GSE). That electrostatic system consists of an infinite conductor sheet laying on the $\mathbb{R}^2$-plane which has two regions that can be thought of as separated conductors (for example, metallic sheets which are very close to each other). The problem is to find the electrostatic field $\boldsymbol{E}(\boldsymbol{r})$ generated by two different potentials: one constant $V_o \neq 0$ potential inside a closed region $\mathscr{A}$ and a grounded $V = 0$ potential in the complementary $\mathbb{R}^2/\mathscr{A}$ region. The GSE problem can be solved from the evaluation of the following Biot-Savart-Like (BSL) expression:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{V_o}{2\pi}\oint_{\partial\mathscr{A}} \frac{d\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3},$$

where $\partial\mathscr{A}$ is the border in between both electrodes. The previous expression becomes the electrostatic analog of the magnetostatic problem in (1). The GSE serves as an ideal model of the gaped Surfaces Electrodes

[1] Being a direct consequence of $\nabla^2 \boldsymbol{v}(\boldsymbol{r}) = \nabla[\nabla \cdot \boldsymbol{v}(\boldsymbol{r})] - \nabla \times [\nabla \times \boldsymbol{v}(\boldsymbol{r})]$.

(SE), which are built technologies of Surface Electrodes[2] [14–23]. Indeed, a growing interest has arisen in the design of SE arrays for the construction of ion traps. In particular, in the localisation of minimums in the electric field $\boldsymbol{E}$ by using external radio frequency fields because ions are trapped there. Advances on expansion techniques to solve BSL integrals are helpful to calculate the set of minimums, which can be constructed with the convergence regions of the expansion solutions.

In the present work, analytic expressions are derived to represent the magnetic field generated by a closed planar electrical wire carrying a steady current. Particularly, the three-dimensional vector field due to a circular-deformed loop with low symmetry. Analytic solutions of the BSL integrals are outstanding for solving steady electric, magnetic or fluid flow problems, as it has been demonstrated in [24, 25] by the authors of the present study [3].

In addition to the Legendre polynomial expansion for the magnetic field of a current in a circular loop, the problem is exactly solvable in the case of the ellipse geometry of the wire [26]. However, there is no exact analytical solution for a generic curved loop. Direct numerical integration of the Biot-Savart formula over C is one possible solution procedure. Correspondingly, numerical and expansion solutions for the perfect circular loop have been obtained in [27–29]. Also, for the electrostatic analog problem of Surface Electrodes, numerical methods have been applied in [24].

There are several benefits of symbolic mathematical expressions of $\boldsymbol{B}(\boldsymbol{r})$ face direct numerical integration that motivates to find analytical expansion formulas. For instance, it is necessary to perform numerical integration for the computation of stream lines or the vector field of $\boldsymbol{B}(\boldsymbol{r})$ at each position in space. Symbolic formulas coming from expansions are direct functions of space. Another scenario where symbolic formulas of $\boldsymbol{B}(\boldsymbol{r})$ can be also useful is in the computation of the force $\boldsymbol{F}(\boldsymbol{r}) = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}(\boldsymbol{r}))$ over a small magnet having dipole moment $\boldsymbol{m}$ in a magnetic field $\boldsymbol{B}$. For this type of problems analytical expansions (as the one in the current study) can be very useful since the spatial derivatives ∇ can be calculated exactly.

The approach in the present work makes use of the Gegenbauer Polynomials as an expansion basis for the three-dimensional magnetic field. This method stands out for the general deformation of the wire, given any deformation function $f(\phi)$ vanishing the symmetry. Hence, the Legendre polynomial expansions of the vector potential $\boldsymbol{A}$ becomes less useful than the present approach since all the three-dimensional components of the vector potential become active and contribute[4][5].

The particular case of the $\mathscr{R}(\phi) = R + H f(\phi)$ loop eventually allows to split the solution into two terms:

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[2]Indeed, the gaped SE solution is essential to describe SE Radio Frequency ion traps: a collection of consecutive SE that have become promising candidates for manufacturing large-scale quantum processors.

[3]For instance, the recent methodology in [25] demonstrates how the electric field of Gaped Surface Electrode systems can be obtained from the weighted average of their gapless counterparts. In that work, it was formally demonstrated that the electric field of the Gapped surface electrode with gap $\mathscr{G}$ is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{V_o}{2\pi}\int_{\mathscr{G}} \frac{\boldsymbol{\mathscr{W}}_\nu(\boldsymbol{r}')d^2\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

and it is analogous to the magnetic field due to a ribbon displayed on $\mathscr{G}$ carrying a current surface density $\boldsymbol{K}$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_o}{4\pi}\int_{\mathscr{G}} \frac{\boldsymbol{K}(\boldsymbol{r}')d^2\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3},$$

where the weight vector $\boldsymbol{\mathscr{W}}_\nu(\boldsymbol{r}')$ depending on the gradient of the electric potential on the $\mathscr{G}$ is the electric counterpart of $\boldsymbol{K}$. Additionally, both fields satisfy analogous continuity conditions $\nabla\cdot\boldsymbol{K}(\boldsymbol{r}) = 0$ for charge conservation, and $\nabla\cdot\boldsymbol{W}(\boldsymbol{r}) = 0$.

[4]For symmetric problems, it could be convenient to find expansions of the vector potential $\boldsymbol{A}$ employing Legendre polynomials since there are components of $\boldsymbol{A}$ that reduce to zero. This is especially the case of the magnetic field for the circular loop.

[5]Additionally, the intermediate step of computing the magnetic field from the rotational operator of the vector potential is not necessary when adopting the Gegenbauer Polynomials.

the solution of the symmetric circular loop and the contribution of the deformation. This is advantageous because the magnetic field of the circular loop can be known exactly from classic results. With this in hand, the method in this document focuses on determining the contribution of the magnetic field from the deformed part. As will be disclosed later, the expansion formulas can be reduced to compute integrals involving the generic $f(\phi)$ function. Moreover, this contribution can be easily handled in some ideal scenarios such as the harmonic deformation, where all the expansion terms can also be found exactly.

The remaining parts of this manuscript are organized as follows. In Section 2, the mathematical setting of the circular-deformed wire problem is recalled. Next, the analytic methodology using the expansion approach with the Gegenbauer Polynomials is presented in Section 3, which ends up giving the magnetic field expression. Illustrative examples of the magnetic field calculation around circular-deformed wires are presented in Section 4. Next, the first-order description for a generic even deformation function is explained in Section 5. Finally, some conclusions are stated in Section 7.

2 The problem

In the present section, the problem set of the circular-deformed wire $\mathscr{R}$ carrying a constant current i is explained. Certainly, the steady magnetic field given by Eq. (1) can be written in terms of the spherical coordinates system (r, θ, ϕ): being r the radial distance, θ the azimuthal angle, and ϕ the polar angle, as follows,

$$B_r(\boldsymbol{r}) = \frac{\mu_o i}{4\pi}\cos\theta \int_0^{2\pi} \frac{\mathscr{R}^2(\phi')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\phi', \quad B_\phi(\boldsymbol{r}) = -\frac{\mu_o i}{4\pi} r\cos\theta \int_0^{2\pi} \frac{\dot{\mathscr{R}}(\phi')\cos(\phi-\phi') + \mathscr{R}(\phi')\sin(\phi-\phi')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\phi',$$

and

$$B_\theta(\boldsymbol{r}) = \frac{\mu_o i}{4\pi}\left[r\int_0^{2\pi} \frac{\mathscr{R}(\phi')\cos(\phi-\phi')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\phi' - \sin\theta \int_0^{2\pi} \frac{\mathscr{R}^2(\phi')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\phi' - r\int_0^{2\pi} \frac{\dot{\mathscr{R}}(\phi')\sin(\phi-\phi')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\phi' \right],$$

where

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{r^2 + \mathscr{R}^2(\phi') - 2r\mathscr{R}(\phi')\sin\theta\cos(\phi-\phi')}. \tag{2}$$

The magnetic field for the simple case of a circular loop $\mathscr{R} = R$ can be written in terms of the complete elliptic integrals of the first and second kind, K, and $\underline{\mathrm{E}}$, respectively. Hence, the magnetic field in terms of the spherical components for the circular loop is given by

$$(B_r(\boldsymbol{r}))_{circle} = \frac{R^2 i}{4\pi}\frac{4\cos\theta}{\mathscr{r}_-(r,\theta)^2\mathscr{r}_+(r,\theta)}\underline{\mathrm{E}}\left[\frac{4rR\sin\theta}{\mathscr{r}_+(r,\theta)^2}\right], \tag{3}$$

$$(B_\theta(\boldsymbol{r}))_{circle} = 2\frac{i}{4\pi}\frac{\csc\theta}{\mathscr{r}_-^2\mathscr{r}_+}\left[(r^2 + R^2\cos(2\theta))\underline{\mathrm{E}}\left(\frac{4rR\sin\theta}{\mathscr{r}_+^2}\right) - \mathscr{r}_-^2 K\left(\frac{4rR\sin\theta}{\mathscr{r}_+^2}\right)\right], \tag{4}$$

$$(B_\phi(\boldsymbol{r}))_{circle} = 0, \tag{5}$$

where $\mathscr{r}_\pm(r,\theta)$ is defined as $\mathscr{r}_\pm(r,\theta) = \sqrt{r^2 + R^2 \pm 2rR\sin\theta}$. It should be noted that the ϕ-component of the magnetic field vanishes because the system is axially symmetric. Nevertheless, the general case $\mathscr{R}(\phi) = R + Hf(\phi)$ of a deformed circular wire $\mathscr{R}(\phi)$ with base radius R and deformation amplitude $\nu = H/R \in (-1,1)$ is more ambitious. Certainly, the magnetic field in the deformed-circular wire is a challenging problem that will be addressed next in the remaining sections of this article. The key to doing this is that the magnetic field can be constructed from $\boldsymbol{B}(\boldsymbol{r}) = (\boldsymbol{B}(\boldsymbol{r}))_{circle} + \boldsymbol{B}^{(\nu)}(\boldsymbol{r})$, with $\boldsymbol{B}^{(\nu)}(\boldsymbol{r})$ the contribution of the deformation.