IOP Publishing Phys. Scr. 100 (2025) 015523 https://doi.org/10.1088/1402-4896/ad98ce

Physica Scripta

RECEIVED
26 April 2024

REVISED
19 November 2024

ACCEPTED FOR PUBLICATION
29 November 2024

PUBLISHED
13 December 2024

PAPER

Modeling of electromagnetic radiation using a dual four-potential representation: from dipole blade radiators to ribbon loop-like antennas

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Keywords: electromagnetic radiation, dual Jefimenko's Equations, blade antennas, gapless and gaped time-dependent surface-electrodes

Abstract

In this paper, we explore classical electromagnetic radiation using a dual four-dimensional potential Θ^{μ} approach. Our focus is on the Planar Dipole Blade Antenna (PDBA), a system consisting of two flat perfect conductive regions on the xy-plane, separated by a gap $\mathcal{G}$, with alternating potentials applied to the conductors. This method emphasizes the use of the scalar magnetic potential $\Psi(\boldsymbol{r}, t)$ and the electric vector potential $\boldsymbol{\Theta}$, which generates the electric field $\boldsymbol{E}(\boldsymbol{r}, t) = \nabla \times \boldsymbol{\Theta}(\boldsymbol{r}, t)$ in free space. These potentials replace the standard magnetic vector potential $\boldsymbol{A}$ and the scalar electric potential $\boldsymbol{\Phi}$ in our analysis. For harmonic radiation, the electromagnetic field can be expressed in terms of the electric vector potential $\boldsymbol{\Theta}(\boldsymbol{r}, t)$. We derive a corresponding retarded vector potential for $\boldsymbol{\Theta}$ in terms of a two-dimensional vector field $W(\boldsymbol{r}, t)$, which flows through the gap region $\mathcal{G}$. This dual analytical approach yields mathematically equivalent expressions for modeling Planar Blade Antennas, analogous to those used for ribbons in the region $\mathcal{G}$, simplifying the mathematical problem. In the gapless limit, this approach reduces the two-dimensional radiator (PDBA) to a one-dimensional wire-loop-like antenna, significantly simplifying the problem's dimensionality. This leads to a dual version of Jefimenko's equations for the electric field, where W behaves like a surface current in the gap region and satisfies a continuity condition. To demonstrate the utility of this approach, we provide an analytical solution for a PDBA with a thin annular gap at low frequency.

1. Introduction

The system of interest in this study consists of two perfect conductive sheets: $\mathcal{A}_{in}$ and $\mathcal{A}_{out}$, both lying on the xy-plane and separated by a gap, defined as follows:

$$\mathcal{G} = \left\{\left(r,\ \phi,\ 0\right):\quad \mathscr{R}(\phi) - \frac{\nu}{2} < r < \mathscr{R}(\phi) + \frac{\nu}{2},\ \forall \quad \phi \in [0,\ 2\pi)\right\}, \tag{1}$$

where $\mathscr{R}(\phi)$, $(\mathscr{R}(\phi) - \nu/2,\ \phi)$, and $(\mathscr{R}(\phi) + \nu/2,\ \phi)$ represent the parametric contours of ∂A, $\partial\mathcal{A}_{in}$, and $\partial\mathcal{A}_{out}$, respectively. These surfaces satisfy the condition $\mathcal{A}_{in} \cup \mathcal{G} \cup \mathcal{A}_{out} = \mathbb{R}^2$.

For clarity, the two-dimensional plane is expressed in Cartesian coordinates as $\mathbb{R}^2 = \{(x,\ y,\ 0)\}$. The finite inner region $\mathcal{A}_{in}$ is characterized by a time-varying scalar potential $\Phi_o(t) = V(t)$, while the outer metallic layer $\mathcal{A}_{out}$ is grounded (see figure 1).

Under these conditions, the system radiates electromagnetic waves into the $\mathbb{R}^3$ space. Our focus is on the radiation in the half-space region $\mathfrak{D} = \{\boldsymbol{r} \in \mathbb{R}^3\colon z > 0\}$, as the system exhibits mirror symmetry with respect to the $z = 0$ plane. This mirror symmetry arises because the electric charges are confined to the conductive layers on the xy-plane.

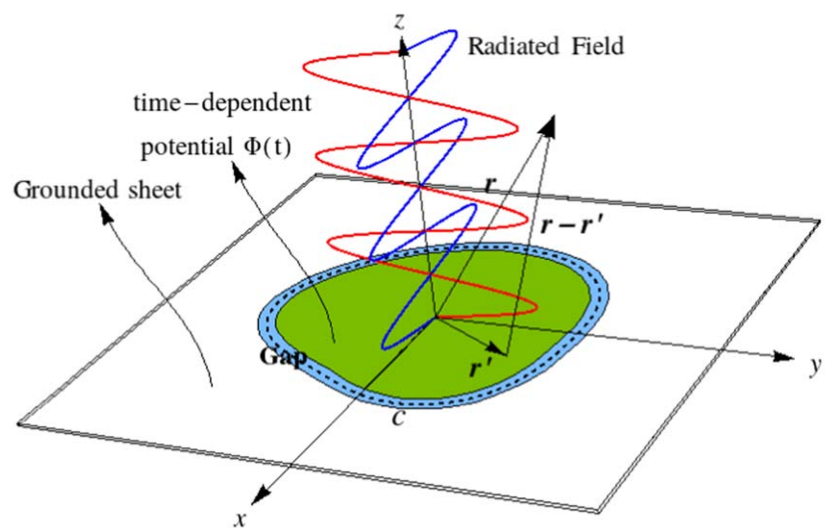

Figure 1. Schematic of the system: Two conductive sheets, $\mathcal{A}_{in}$ and $\mathcal{A}_{out}$, separated by a gap $\mathcal{G}$. The inner sheet $\mathcal{A}_{in}$ is subjected to a time-dependent potential, $V(t)$, while the outer sheet $\mathcal{A}_{out}$ is grounded.

The system of interest is configured as a Planar Dipole Blade Antenna (PDBA) radiating in free space. This technology is crucial for various technical applications, including communications, navigation, aeronautics [1], and astronomy [2, 3]. Theoretical modeling of antenna devices has garnered significant interest, particularly in wire-antenna radiation setups, where wires are driven by alternating voltage sources. Other key studies focus on circular loop antennas [4–10], utilizing techniques such as the method of moments [11, 12] and basis function expansions [13]. In general, wire antennas are mathematically simpler than PDBAs due to the spatial localization of currents. Other key studies focus on the design and fabrication of antennas, such as the Pattern Director Antenna (PDA), which simplifies mode conversion while achieving high directivity [14] and the Smile-like Mode Converter Antenna (SMCA), designed for high-power applications to enhance the efficiency and power capacity of the device [15].

Dielectric antennas have attracted attention for their high radiation efficiency, compact size, and lightweight structure. Unlike dipole blade antennas, they use dielectric materials to guide and radiate electromagnetic waves. An example is the dielectric resonator antenna (DRA), which enhances radiation by exciting resonance modes within the material. Recent innovations, like printable planar dielectric antennas, achieve high efficiency and broadside patterns with low cross-polarization at Ku-band frequencies [16]. DRAs have advanced in multi-resonance, broadband, and ultra-wideband designs, improving bandwidth and stability [17]. Despite structural differences from dipole-blade antennas, dielectric antennas offer valuable advantages across modern applications [18]. Nano-antennas, operating at optical frequencies, extend this concept in nanotechnology to interact with light at subwavelength scales. Unlike traditional antennas, they rely on plasmonic [19] or dielectric materials to enable strong light-matter interactions, making them ideal for applications in sensing and optical communications. Additionally, dielectric nano-antennas provide enhanced field confinement and avoid metallic losses [20]. These capabilities at the intersection of photonics and electronics contrast to both dipole-blade and dielectric antennas [21].

The main goal of this paper is to model PDBA radiation by formulating the dynamic phenomena through the vector potential $\boldsymbol{\Theta}(\mathbf{r}, t)$, where the electric field is expressed as $\boldsymbol{E} = \nabla \times \boldsymbol{\Theta}$. Using this approach, we demonstrate that PDBAs, such as the one depicted in figure 1, can be effectively mapped to a wire loop-like antenna counterpart (see figure 2) in the half-space $\mathcal{D}$. This is a generalization of the formalism developed for Surface Electrodes (SE) [22–24], which represents the electrostatic version of the system shown in figure 1. In that context, the electric field is represented by a Biot-Savart-like integral, associating a vector potential with the electric field instead of a scalar potential.

The concept of the electric vector potential is not new. In magnetostatics, associating a scalar magnetic potential to the magnetic field is standard practice. However, associating a vector potential to the electric field is less common but applicable in various scenarios [25–28]. A classic example is the study of electromagnetic radiation in the Coaxial Line onto a Ground Plane (CLGP) system. There, the electric field $\boldsymbol{E}$ in the transition line is analogous to that of an infinitely long cylindrical capacitor. The radiated field of the CLGP can be approximated by a *magnetic current field* circulating around the aperture (the coaxial opening), following the same profile as $\boldsymbol{E}$ [29, 30]. The

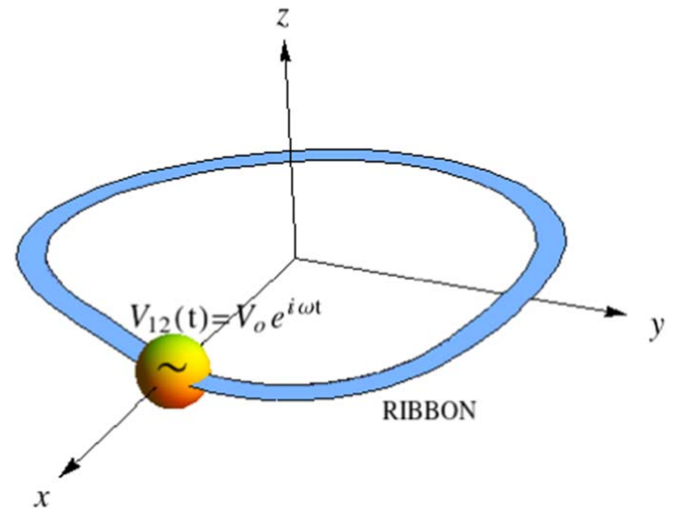

Figure 2. Ribbon fed by a time-dependent voltage.

approach of identifying two different systems that share the same mathematical structure, allowing their solutions to be interchanged, is commonly referred to as the duality theorem (see [31], Chapter 3, Section 3.7).

The goal of this work is to rigorously establish the mathematical equivalence between the PDBA and a ribbon-like antenna subjected to a specific surface current density. This is particularly important since the fields in the gap region are generally unknown *a priori*. For the magnetostatic counterpart in the low-frequency limit, the electric current has an electrostatic counterpart: the Surface Electrode (SE), assuming the same gap and ribbon geometries. To achieve this, we must properly define the Green's function for the problem, ensuring the correct inclusion of the system's boundary conditions. The contribution of the paper is the formal demonstration of the mathematical equivalency in the specific antenna setting. The demonstration is required to avoid mismatching and in general, it is laborious since it is necessary to build properly the Green's function of the dual potentials to fit correctly the boundary conditions, as it has been done for the PDBA. The paper also exposes a numeric-analytic strategy to approximate $\mathcal{W}(\boldsymbol{r}, t)$ in the gap region.

This document is organized as follows. In Section 2, we describe the electric vector potential $\boldsymbol{\Theta}(\boldsymbol{r}, t)$ and the dual 4-electromagnetic potential. By the end of that section, we derive the wave equation in free space for both the electric vector potential and the magnetic scalar potential. Section 3 is dedicated to finding an integral expression for the electric vector potential by solving the wave equation. In this section, we also discuss the appropriate boundary conditions on the metallic sheets and the gap. In Section 4, the radiated electric field is derived, and we establish a correspondence with Jefimenko's equations. We demonstrate that PDBA radiation is mathematically equivalent to the radiation from a ribbon loop-like structure of alternating current located in the PDBA gap region. Section 5 provides expressions for the electromagnetic field in the specific case of harmonic radiation. An example applying the analytic strategy to the low-frequency radiation of a circular planar dipole blade antenna is presented in Section 6. In Section 6.1, we test the consistency of the system with dipole radiation in the far-field limit. Additionally, we present computations of the fields near the gap as a function of the gap thickness at low frequencies. Finally, the conclusions are summarized in Section 7.

2. The electromagnetic dual 4-potential vector representation

In the current section we begin by outlining the strong form of the problem, which provides a direct description of the field equations. This is followed by a brief review of the standard electromagnetic 4-potential vector, which allows for the transformation of the strong form into an integral form involving the field sources: charges and currents.

The core of this work focuses on the application of a dual electromagnetic 4-potential vector representation. Thus, the treatment of the wave propagation problem is discussed in detail in the remaining part of this section.

2.1. Standard electromagnetic 4-potential vector

A standard approach to derive the electric and magnetic fields is through the electromagnetic 4-potential:

$$A^{\mu}(\boldsymbol{r},\, t) := \left(\frac{1}{c}\Phi(\boldsymbol{r},\, t),\, \boldsymbol{A}(\boldsymbol{r},\, t)\right), \tag{2}$$

which satisfies the following vector identities:

$$\boldsymbol{E}(\boldsymbol{r},\, t) = -\nabla\ \ \Phi(\boldsymbol{r},\, t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},\, t)}{\partial t} \text{ and } \boldsymbol{B}(\boldsymbol{r},\, t) = \nabla\times\boldsymbol{A}(\boldsymbol{r},\, t),$$

where Φ is the scalar electric potential and $\boldsymbol{A}$ is the magnetic vector potential.

In general, these fields have integral solutions that account for their spatial sources:

$$\Phi(\boldsymbol{r},\, t) = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3}\frac{\rho(\boldsymbol{r}',\, t_r)}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d^3\boldsymbol{r}', \tag{3}$$

$$\boldsymbol{A}(\boldsymbol{r},\, t) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\frac{\boldsymbol{J}(\boldsymbol{r}',\, t_r)}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d^3\boldsymbol{r}', \tag{4}$$

with t_r being the retarded time.

These integral formulas are useful for wire-radiation problems, including low-frequency radiation, where the charge density and current density can be assumed to be uniformly distributed. In the low-frequency case, $\boldsymbol{J}$ is typically well-localized along the infinitely thin wire path. However, the application of equations (3) and (4) becomes more challenging in surface radiation scenarios. This difficulty arises because sources, both *charges and currents*, are generally not uniformly distributed over the antenna surfaces.

Even in the electrostatic versions of the surface system [22, 24], the distribution of surface charge density on the metallic sheets strongly depends on the gap geometry. Therefore, the stationary form of equation (3) is not applicable, as the charge density sources are *a priori* unknown.

2.2. Dual electromagnetic 4-potential vector

The mathematical electrostatic problem can be simplified by associating a *vector potential* $\boldsymbol{\Theta}$ with the electric field instead of the scalar potential Φ. Our goal here is to generalize the electrostatic strategy introduced in [22, 24] to time-dependent radiation problems.

In free-charge space, where both the current density $\boldsymbol{J}(\boldsymbol{r}, t) = 0$ and the charge density $\rho(\boldsymbol{r}, t) = 0$, the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ becomes divergenceless, i.e., $\nabla\cdot\boldsymbol{E}(\boldsymbol{r}, t) = 0$, allowing us to express it as:

$$\boldsymbol{E}(\boldsymbol{r},\, t) = \nabla\times\boldsymbol{\Theta}(\boldsymbol{r},\, t), \tag{5}$$

where $\boldsymbol{\Theta}(\boldsymbol{r}, t)$ is any vector potential satisfying this identity. This *electric vector potential* plays a role analogous to the standard magnetic vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$, which is related to the magnetic field via the curl operation $\boldsymbol{B}(\boldsymbol{r}, t) = \nabla\times\boldsymbol{A}(\boldsymbol{r}, t)$. We can combine equation (5) with Ampere-Maxwell's law

$$\nabla\,\times\,\boldsymbol{B}(\boldsymbol{r},\, t) = \mu_0\boldsymbol{J}(\boldsymbol{r},\, t) + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}(\boldsymbol{r},\, t)}{\partial t}$$

in vacuum ($\boldsymbol{J} = 0$), leading to:

$$\nabla\,\times\,\boldsymbol{B}(\boldsymbol{r},\, t) = \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\,\times\,\boldsymbol{\Theta}(\boldsymbol{r},\, t))\;\therefore\;\nabla\,\times\left(\boldsymbol{B}(\boldsymbol{r},\, t) - \frac{\partial}{\partial t}\frac{\boldsymbol{\Theta}(\boldsymbol{r},\, t)}{c^2}\right) = 0.$$

Since the curl of the gradient of any scalar field is zero, we can express the term inside the parentheses as the gradient of an arbitrary scalar field, say $\nabla(-\mu_0\Psi(\boldsymbol{r}, t))$. Here, $\Psi(\boldsymbol{r}, t)$ is identified as the *magnetic scalar potential*, a convenient choice that maintains dimensional consistency. Thus, the magnetic field takes the form:

$$\boldsymbol{B}(\boldsymbol{r},\, t) = -\mu_0\,\nabla\Psi(\boldsymbol{r},\, t) + \frac{\partial}{\partial t}\frac{\boldsymbol{\Theta}(\boldsymbol{r},\, t)}{c^2}. \tag{6}$$

Given equations (5) and (6), it is useful to compose a dual 4-potential vector by organizing the terms as:

$$\boldsymbol{\Theta}^{\mu}(\boldsymbol{r},\, t) := \left(a_0\mu_0\Psi(\boldsymbol{r}),\, a_1\frac{\Theta_x(\boldsymbol{r},\, t)}{c^2},\, a_1\frac{\Theta_y(\boldsymbol{r},\, t)}{c^2},\, a_1\frac{\Theta_z(\boldsymbol{r},\, t)}{c^2}\right),$$

where the first term corresponds to the magnetic scalar potential and the subsequent terms include the components of the electric vector potential. Here, a_0 and a_1 are constants specific to the problem. We adopt physical coordinates as $\{x^{\mu}\}_{\mu=0,\ldots,3} = \{ct,\, x,\, y,\, z\}$.

Since $\boldsymbol{\Theta}'(\boldsymbol{r},\, t) = \boldsymbol{\Theta}(\boldsymbol{r},\, t) + \nabla\lambda(\boldsymbol{r},\, t)$, where λ is an arbitrary scalar function, the vector potential in equation (5) is not unique. Therefore, a divergence-free condition must be imposed on the 4-potential vector, taking the form:

$$\partial_\mu \boldsymbol{\Theta}^\mu(\boldsymbol{r}, t) = \boldsymbol{\Theta}^\mu_{,\mu}(\boldsymbol{r}, t) = 0,$$

which resembles a Lorentz gauge-like condition. Here, we adopt the Einstein summation convention for repeated indices, with ∂_μ denoting a covariant derivative. [4]

Finally, we demonstrate that the dual 4-potential vector is invariant under space-time transformations, as detailed in Appendix A.

2.2.1. D'Alembert equation for the magnetic scalar potential

Rearranging the summation result, we find the following relation between the temporal term and the spatial divergence:

$$\nabla \cdot \left(\frac{1}{c^2}\boldsymbol{\Theta}(\boldsymbol{r}, t)\right) = -\frac{1}{a_1}\frac{\partial}{\partial(ct)}(a_0\mu_0\Psi(\boldsymbol{r}, t)). \tag{7}$$

Additionally, we can apply Gauss's law for the magnetic field to equation (6):

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}, t) = -\mu_0\nabla^2 \ \Psi(\boldsymbol{r}, t) + \frac{\partial}{\partial t}\frac{\nabla \cdot \boldsymbol{\Theta}(\boldsymbol{r}, t)}{c^2} = 0,$$

to obtain the constants a_0 and a_1. By combining this with equation (7), we have:

$$\nabla^2(\mu_0\Psi(\boldsymbol{r}, t)) + \frac{\partial}{\partial t}\left[\frac{1}{a_1}\frac{\partial}{\partial(ct)}(a_0\mu_0\Psi(\boldsymbol{r}, t))\right] = 0.$$

Assuming $a_0 = -a_1/c$, this expression simplifies to:

$$\Box^2\Psi(\boldsymbol{r}, t) = 0, \quad \text{where} \quad \Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \tag{8}$$

with $\Box^2$ representing the D'Alembertian operator.

We are still free to choose the value of one of the constants. If we set $a_1 = -1$, then $a_0 = 1/c$, and the dual version of the electromagnetic potential takes the form:

$$\boldsymbol{\Theta}^\mu(\boldsymbol{r}, t) := \left(\frac{\mu_0}{c}\Psi(\boldsymbol{r}, t), -\frac{\boldsymbol{\Theta}(\boldsymbol{r}, t)}{c^2}\right), \tag{9}$$

under the following Lorentz gauge-like condition:

$$\nabla \cdot \boldsymbol{\Theta}(\boldsymbol{r}, t) - \mu_0\frac{\partial\Psi(\boldsymbol{r}, t)}{\partial t} = 0. \tag{10}$$

2.2.2. D'Alambert equation for the dual 4-potential vector

Now, we can substitute equation (5) into Faraday's law:

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = \nabla \times \nabla \times \boldsymbol{\Theta}(\boldsymbol{r}, t) = -\frac{\partial\boldsymbol{B}(\boldsymbol{r}, t)}{\partial t}.$$

By applying the vector calculus identity $\nabla^2\boldsymbol{F}(\boldsymbol{r}) = \nabla(\nabla \cdot \boldsymbol{F}(\boldsymbol{r})) - \nabla \times \nabla \times \boldsymbol{F}(\boldsymbol{r})$ for any vector field $\boldsymbol{F}(\boldsymbol{r})$, we obtain:

$$-\nabla^2\boldsymbol{\Theta}(\boldsymbol{r}, t) + \nabla(\nabla \cdot \boldsymbol{\Theta}(\boldsymbol{r}, t)) = -\frac{\partial\boldsymbol{B}(\boldsymbol{r}, t)}{\partial t}.$$

Next, using the Lorentz-like gauge condition from equation (7) with $(a_0, a_1) = (1/c, -1)$, we simplify the equation to:

$$-\nabla^2\boldsymbol{\Theta}(\boldsymbol{r}, t) + \nabla\left(-\mu_0\frac{\partial}{\partial t}\Psi(\boldsymbol{r}, t)\right) = -\frac{\partial\boldsymbol{B}(\boldsymbol{r}, t)}{\partial t}.$$

This result can be further rearranged as:

$$-\nabla^2\boldsymbol{\Theta}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}(\boldsymbol{B}(\boldsymbol{r}, t) + \mu_0\nabla\Psi(\boldsymbol{r}, t)) = -\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\Theta}(\boldsymbol{r}, t)}{c^2}\right),$$

where we have used the relation from equation (6). Thus, we obtain the wave equation for the dual 4-potential vector:

[4] This is essential for selecting an appropriate gauge condition. For time-steady problems, this condition reduces to $\nabla \cdot \boldsymbol{\Theta} = 0$, similar to the Coulomb gauge. This gauge is useful for modeling electrostatic systems via the electric vector potential, as shown in [22, 24] for surface electrodes and Coulomb two-dimensional gases.