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Variational multiscale approximation of the one-dimensional forced Burgers equation: The role of orthogonal subgrid scales in turbulence modeling

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Summary

A numerical approximation for the one-dimensional Burgers equation is proposed by means of the orthogonal subgrid scales–variational multiscale (OSGS-VMS) method. We evaluate the role of the variational subscales in describing the Burgers “turbulence” phenomena. Particularly, we seek to clarify the interaction between the subscales and the resolved scales when the former are defined to be orthogonal to the finite-dimensional space. Direct numerical simulation is used to evaluate the resulting OSGS-VMS energy spectra. The comparison against a large eddy simulation model is presented for numerical discretizations in which the grid is not capable of solving the small scales. An accurate approximation to the phenomena of turbulence is obtained with the addition of the purely dissipative numerical terms given by the OSGS-VMS method without any modification of the continuous problem.

KEYWORDS

Burgers equation, direct numerical simulation, large eddy simulation, turbulence, orthogonal subgrid scales, variational multiscale method

1 | INTRODUCTION

The Burgers equation is a simplified model of the equations that govern fluid flow, and in spite of its simplicity, it shares some features of the nonlinear dynamics present in the Navier-Stokes equations. Burgers[1] attempted unsuccessfully to arrive at a statistical theory of turbulent fluid motion, but he was able to clarify the interaction between transient, dissipative, and nonlinear terms in an extremely simplified equation of motion. His contribution still represents a significant achievement when one aims to describe turbulence: Burgers turbulence exhibits an energy cascade that is similar to that encountered in fluid flow turbulence governed by the Navier-Stokes equations, containing also an inertial and a dissipation range.

Analytic solutions are known for the forced Burgers equation: When the nonlinear term is dominant it leads to the formation of shocks, and the solution exhibits an energy spectrum that behaves as k^{-2} in the inertial range, k being the wavenumber. A complete review of the analytical solutions by Hopf[2] and Cole,[3] including some of the mathematical and computational framework for the viscous form of the time-dependent Burgers equation, has been reported by Surana et al.[4] Nevertheless, the important difference between the Navier-Stokes and the Burgers dynamics is that the smallest scales that dissipate the energy of the flow in the Burgers equation do not refer to the smallest eddies, but to the shock thickness instead.

Numerically resolving every scale of the solution is what is referred as the direct numerical simulation (DNS). Most of the numerical approaches that aim to correctly describe turbulent flows can be prohibitively expensive for nearly all systems with complex geometries, because the discrete meshes of the approximation (or the number of basis functions in the approximation space) need to be refined up to a power of the Reynolds number. Hence, it is presently impossible to simulate flows at high Reynolds numbers using DNS, and in most of the practical fluid dynamics applications turbulence phenomena are modeled somehow.

Several alternatives have raised in the turbulence modeling community, being large eddy simulation (LES) one of the most robust and prevailing approaches. The basic idea of this method is to filter the fluid equations in space and time, simulating only the filtered large scales, while the smallest (and most expensive to compute) scales are incorporated into the overall solution by including a modeling term in the filtered equations. The filtered scales are called the resolved scales of the flow, and the scales below the resolved scales are called the subscales. However, filtering the equations results in an unclosed term involving the subscales that cannot be determined from the resolved quantities, so that it needs to be calculated by assuming some *a priori* properties of the flow. In particular, eddy viscosity type models, such as the Smagorinsky model[5] and the Germano model,[6] apply proportionality constants that must be set empirically *a priori*, and this makes them incomplete models. Some other authors[7,8] approximated the problem by decomposing the resolved and unresolved scales into deterministic and stochastic components, in which the subscales are calculated by using a stochastic estimation. And yet as reviewed by Das and Moser,[9] all these models fail to accurately represent inhomogeneous flows.

Another method that separates the solution into resolved and unresolved scales is the variational multiscale (VMS) method.[10] Alternatively to the LES approach, this method splits the solution in order to stabilize the numerical approximation of the problem. Fluid flow equations have been traditionally stabilized by using the family of VMS methods, including the algebraic subgrid scales and the orthogonal subgrid scales (OSGS) methods (like in the work of Codina[11]). The original idea of using the multiscale formulation with local approximation to the fine scales to compute turbulent flows was introduced by Codina[12] and later discussed by Hughes et al.[13] The illustration that the VMS formulation works as a turbulent model, which depends on the validity of the approximation made to derive the evolution equation for the unresolved scales, was elaborated in previous studies.[14,15] Some other authors proposed to further split the resolved scales into coarse and fine scales and to adopt the hypothesis that the subscales do not affect the coarse resolved scales. Among these methods we can distinguish the works of Wasberg et al[16] and Gravemeier and Wall,[17] in which the effect of the unresolved scales is introduced only into the fine resolved scales. More recently, some authors[18-22] have directly applied the VMS method in order to solve the Navier-Stokes turbulence, in what are called implicit LES (ILES) techniques. This approach accurately solves turbulence, relying on the addition of dissipative numerical terms solely, and without any modification of the continuous problem.

In this work we aim to apply the OSGS-VMS method to the numerical approximation of the one-dimensional forced Burgers equation. In the spirit of ILES methods, we aim to clarify the mechanisms through which the resolved and unresolved scales interact with each other and remark about the role of the OSGS in modeling Burgers turbulence. The proposed numerical approximation exploits the Fourier transform of the Burgers equation, following the work of Wang and Oberai,[23] in which the scale dependence on the numerical dissipation introduced by the subscales is clarified. Complementary to that approach, we include the OSGS into the resolved scales equation by means of the adjoint of the nonlinear operator applied to the test function, so that both terms associated with the Cross and Reynolds stresses are accounted for. Solving the subscales in a separated equation (subscales equation) leads to the inclusion of a Leonard stress term. In addition, we propose to calculate subscales in terms of the resolved scales by defining *a priori* the space where the subscales exist (as the orthogonal space to the finite-dimensional resolved space), and approximating the nonlinear operator associated to the Burgers problem. The scale dependence of the introduced numerical dissipation terms is studied by considering the triadic interactions among the resolved scales and the subscales. More precisely, terms associated with the Leonard, Cross, and Reynolds stresses (involving resolved and unresolved scales) allow forward and backward scattering. Contrary to the work of Li and Wang,[24] we are not interested in transient turbulent solutions. Instead, we test the OSGS-VMS method in contrast to the static Smagorinksy LES model and present some conclusions about the behavior of the orthogonal subscales in the numerical approximation of the Burgers turbulence phenomena.

The outline of this article is organized as follows. In Section 2 we define the Burgers equation and the VMS form of the problem. The numerical approximation used to solve the problem is presented in Section 3. Section 4 shows the numerical results for a test example. OSGS-VMS results are compared both to DNS and LES simulations at the end of that section. Finally, conclusions are stated in Section 5.

2 | PROBLEM DEFINITION

In this section we define the VMS approximation of the Burgers problem. First, we recall the one-dimensional forced Burgers equation in the physical space and derive the Galerkin form of the problem. Then, we apply the VMS framework and present the equations for both the resolved scales and the subscales.

2.1 | Burgers equation

The one-dimensional forced Burgers equation[25] consists of finding a scalar function $u(x, t)$ of position $x \in \Omega = (0, 2\pi)$ and of time $t \geqslant 0$ such that, given a forcing term $f(x, t)$,

$$\partial_t u + u\partial_x u - \nu\partial_{xx} u = f, \qquad x \in (0, 2\pi), \quad t > 0, \tag{1}$$

with an initial condition $u(x, 0) = u^0(x)$ in Ω and periodic boundary conditions at $x = 0$ and $x = 2\pi$, $t \geqslant 0$. The diffusivity coefficient ν is considered as homogeneous and constant.

The Burgers equation shares some of the properties of the Navier-Stokes equations. Apart from the temporal derivative, the left-hand side (LHS) contains both a nonlinear and a linear term, associated with convection and diffusion, respectively. The interaction between the dissipation given by the diffusive term (containing second order derivatives) and the convection given by the nonlinear inertial term also appears in the incompressible Navier-Stokes equations.

For convenience Equation 1 can be written in the form

$$\partial_t u + \mathscr{L}(u; u) = f, \qquad x \in (0, 2\pi), \quad t > 0, \tag{2}$$

by introducing the bilinear operator $\mathscr{L}(u; v) = u\partial_x v - \nu\partial_{xx} v$. For smooth solutions, $\mathscr{L}(u; u) = u\partial_x u - \nu\partial_{xx} u = \partial_x \left(\frac{1}{2}u^2 - \nu\partial_x u\right)$. The latter is the form that admits physically meaningful solutions when they develop discontinuities; it is known as the conservation form.

Let $\mathscr{W} = H^1_{\text{per}}(\Omega)$ be the subspace of periodic functions in $H^1(\tilde{u})$, and let us write $(f, g) = \int_\Omega fg$ for any two functions f and g. Introducing the form

$$A(u; w, v) := -\frac{1}{2}(u\partial_x w, v) + \nu(\partial_x w, \partial_x v), \tag{3}$$

the variational form of the problem can be written as: find $u : \mathbb{R}^+ \to \mathscr{W}$ such that

$$(w, \partial_t u) + A(u; w, u) = (w, f), \quad t > 0, \tag{4}$$

$$(w, u) = (w, u^0), \quad t = 0, \tag{5}$$

for all $w \in \mathscr{W}$.

2.2 | Variational multiscale method

Let us consider a finite-dimensional subspace $\mathscr{W}_N \subset \mathscr{W}$, of dimension N, which approximates $\mathscr{W}$ as $N \to \infty$. The Galerkin approximation to problem (4) and (5) can be stated as follows: find $u_N : \mathbb{R}^+ \to \mathscr{W}_N$ such that

$$(w_N, \partial_t u_N) + A(u_N; w_N, u_N) = (w_N, f), \quad t > 0, \tag{6}$$

$$(w_N, u_N) = (w_N, u^0), \quad t = 0, \tag{7}$$

for all $w_N \in \mathscr{W}_N$. This approximation suffers from instability problems, which vary according to the way to construct $\mathscr{W}_N$, but which are in any case due to the convective property of the nonlinear term.

The idea of the VMS method is to decompose the space of the unknown into a finite-dimensional space $\mathscr{W}_N$, and an infinite-dimensional one, $\tilde{\mathscr{W}}$, so that $\mathscr{W} = \mathscr{W}_N \oplus \tilde{\mathscr{W}}$. The unknown and the test functions are accordingly split as $u = u_N + \tilde{u}$ and $w = w_N + \tilde{w}$, respectively. We shall refer to functions in $\mathscr{W}_N$ as the *large* or *resolved* scales and to functions in $\tilde{\mathscr{W}}$ as the *subscales* or *unresolved* scales.

Equation 4 can be equivalently written as the system of equations

$$(w_N, \partial_t u) + A(u; w_N, u) = (w_N, f), \quad \text{for all } w_N \in \mathscr{W}_N, \quad t > 0, \tag{8}$$

$$(\tilde{w}, \partial_t u) + A(u; \tilde{w}, u) = (\tilde{w}, f), \quad \text{for all } \tilde{w} \in \tilde{\mathscr{W}}, \quad t > 0, \tag{9}$$

and likewise for the initial condition, Equation 5. The second term in the LHS of Equation 8 can be split as

$$A(u;w_N,u) = A(u_N;w_N,u_N) + A(u_N;w_N,\tilde{u}) + A(\tilde{u};w_N,u_N) + A(\tilde{u};w_N,\tilde{u}). \tag{10}$$

We may define

$$A(u_N;w_N,u_N) \quad \text{Galerkin terms} \tag{11}$$

$$A(u_N;w_N,\tilde{u}) + A(\tilde{u};w_N,u_N) \quad \text{Nonlinear Cross terms} \tag{12}$$

$$A(\tilde{u};w_N,\tilde{u}) \quad \text{Nonlinear contribution of the unresolved scales} \tag{13}$$

If $\mathscr{W}_N$ is made of smooth functions, as it is the case considered below, it is readily seen that the equation for the unresolved scales (9) can be written as

$$(\tilde{w},\partial_t\tilde{u}) + (\tilde{w},\mathscr{L}(u;\tilde{u})) = (\tilde{w},f - \partial_t u_N - \mathscr{L}(u;u_N)), \quad \text{for all } \tilde{w} \in \tilde{\mathscr{W}}, \quad t > 0. \tag{14}$$

The objective of the VMS method is to approximate the subscales in terms of the resolved scales, to end up with a problem for the resolved scales alone. The key point is the approximation of $\mathscr{L}(u;\tilde{u})$ to make Equation 14 easily solvable. We will not describe the motivation, which can be based on a Fourier analysis of the problem, but the approximation that we consider is (see Codina et al[26])

$$\mathscr{L}(u;\tilde{u}) \approx \tau^{-1}(u)\tilde{u}, \tag{15}$$

where $\tau(u) > 0$ is a numerical parameter of the formulation, the expression of which is given later for a particular approximation. The objective of (15) is not to provide an accurate pointwise approximation to $\tilde{u}$, but to capture its effect on the resolved scales.

The residual of the resolved scales is defined as $R_N(u;u_N) = f - \partial_t u_N - \mathscr{L}(u;u_N)$. If approximation (15) is inserted into (14), one obtains a nonlinear problem for $\tilde{u}$. Even though it is possible to deal with this nonlinearity (see Codina et al[26]), we will consider that the unresolved scales are smaller than the resolved ones, so that

$$\tau(u) \approx \tau(u_N), \quad R_N(u;u_N) \approx R_N(u_N;u_N). \tag{16}$$

These approximations and (15) allow us to write the approximate equation for the subscales as

$$\left(\tilde{w},\partial_t\tilde{u} + \tau^{-1}(u_N)\tilde{u}\right) = (\tilde{w},R_N(u_N;u_N)), \qquad \text{for all} \quad \tilde{w} \in \tilde{\mathscr{W}}, \quad t > 0. \tag{17}$$

Even though we have not distinguished them, the subscales solution of (17) are approximate, whereas those solution of (14) are exact.

The subscales space is generally defined in two different ways. The first, and the most common approach, is to define it as the space of residuals of the resolved scales, and therefore to consider that

$$\partial_t\tilde{u} + \tau^{-1}(u_N)\tilde{u} = R_N(u_N;u_N), \qquad t > 0.$$

This is what we call algebraic subgrid scales method in finite elements. The second approach is the OSGS method, which defines the subscales space as the space orthogonal to the resolved scales space. If P_N is the L^2-projection onto $\mathscr{W}_N$ and $P_N^{\perp} = I - P_N$, I being the identity, then the equation for the unresolved scales in this case is

$$\partial_t\tilde{u} + \tau^{-1}(u_N)\tilde{u} = P_N^{\perp}[R_N(u_N;u_N)], \quad t > 0.$$

In the spectral approximation described next, we shall see that the OSGS method is in fact the natural approach, because the basis functions are mutually L^2-orthogonal (and also H^1-orthogonal). In this case, we thus have that $\mathscr{W} = \mathscr{W}_N \oplus \mathscr{W}_N^{\perp}$, with $\mathscr{W}_N^{\perp} = \tilde{\mathscr{W}}$.

3 | SOLUTION METHOD

In the previous section, we have defined the variational formulation for each scale of the Burgers problem. The 2 subproblems (8) and (17) are now numerically approximated by transforming them into the Fourier space or, equivalently, by using a spectral Fourier method. Considering $\tilde{u} = 0$ would reduce the problem to be solved to (6) and (7), ie, the Galerkin-Fourier method. It is well known that this approximation of the Burgers equation suffers from numerical instabilities. In this section we recall the importance of the numerical diffusion introduced by the VMS formulation, not only to prevent spurious oscillations in the numerical solution but also to model the "turbulence" phenomena. Next, we derive the energy equation in Fourier space. Finally, the time-integration scheme is described.

3.1 | Discrete Burgers equation

Let us consider N even and define the discrete space where the solution is sought as

$$\mathscr{W}_N := \text{span}\,\{e^{ikx} : k \in [-N/2, N/2-1] \subset \mathbb{N}\},$$

ie, the space spanned by N Fourier modes, k being the wavenumber. Let us write $(f,g) = \int_\Omega \bar{f} g$ for the L_2 inner product over complex-valued functions f and g, by denoting the complex conjugate with an overline. The approximate unknown can thus be expressed as

$$u_N(x,t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t) e^{ikx},$$

where $\hat{u}_k(t) := \left(u_N(x,t), e^{ikx}\right)$. We recall the orthogonality relation $\left(e^{ikx}, e^{ilx}\right) = 2\pi\delta_{kl}$, where δ_{kl} is the Kronecker delta. The notation $\sum_{k=-N/2}^{N/2-1} \hat{u}_k(t)e^{ikx} := \sum_k \hat{u}_k e^{ikx}$ will be used in the following.

To obtain the discrete weak form of the problem, let us test the differential Equation 8 with the basis functions of $\mathscr{W}_N$. If $\hat{f}_l$, $l \in \mathbb{N}$, are the Fourier coefficients of the forcing term f, we have that $\left(e^{ikx}, \sum_l \hat{f}_l e^{ilx}\right) = 2\pi\hat{f}_k$. For the transient and diffusive terms we have

$$\left(e^{ikx}, \partial_t \sum_l \hat{u}_l e^{ilx}\right) = 2\pi\partial_t\hat{u}_k, \quad \left(\nu\partial_x e^{ikx}, \partial_x \sum_l \hat{u}_l e^{ilx}\right) = -2\pi\nu k^2\hat{u}_k,$$

for all $k, l \in [-N/2, N/2-1]$ and $t > 0$. Moreover, the nonlinear Galerkin term results in

$$\left(\frac{1}{2}\sum_q \hat{u}_q e^{iqx}\partial_x e^{ikx}, \sum_l \hat{u}_l e^{ilx}\right) = -\pi i k \sum_{q+l=k} \hat{u}_q\hat{u}_l, \tag{18}$$

for all $k, q, l \in [-N/2, N/2-1]$ and $t > 0$. The remaining terms of Equation 10, which depend on the subscales, are developed next. Note first that the transient and diffusive terms belong to $\mathscr{W}_N$, and we may assume that f also belongs to this space without losing accuracy, because the order of the error made will be the same as that of approximating $\mathscr{W}$ by $\mathscr{W}_N$. Therefore, if the subscales are defined to belong to the space orthogonal to the finite-dimensional resolved space, then the subscales equation can be expressed as

$$\left(\tilde{w}, \partial_t\tilde{u} + \tau^{-1}(u_N)\tilde{u}\right) = \left(\tilde{w}, -u_N\partial_x u_N + P_N(u_N\partial_x u_N)\right), \qquad \text{for all } \tilde{w} \in \tilde{\mathscr{W}}. \tag{19}$$

Now, the term $u_N\partial_x u_N$ can be developed as

$$u_N\partial_x u_N = \sum_q e^{iqx}\hat{u}_q \sum_l il e^{ilx}\hat{u}_l = \sum_{q,l} il e^{i(q+l)x}\hat{u}_q\hat{u}_l,$$

for all $q, l \in [-N/2, N/2-1]$. This term belongs to the space span$\{e^{irx} : r \in [-N, N-2]\}$. The projection onto the space of resolvable scales, P_N, can be obtained by considering in the sum above only the wavenumbers q, l, that satisfy $q + l \in [-N/2, N/2-1]$. Consequently, the orthogonal subscales belong to the Fourier space $\tilde{\mathscr{W}}$, given by

$$\tilde{\mathscr{W}} := \text{span}\{e^{irx} : r \in [-N, N-2] \setminus [-N/2, N/2-1]\}.$$

With this definition, taking $\tilde{w} = e^{irx}$ in (19), we obtain the equation for the subscales

$$\partial_t\hat{\tilde{u}}_r + \tau^{-1}(u_N)\hat{\tilde{u}}_r = \sum_{q+l=r} il\hat{u}_q\hat{u}_l, \tag{20}$$

for all $q, l \in [-N/2, N/2-1]$, $r \in [-N, N-2] \setminus [-N/2, N/2-1]$, and $t > 0$.

To close the subscales calculation, the approximation of the Burgers nonlinear operator (15) is needed. The key point in the design of the VMS formulation is the construction of τ. In this work we use the approximation of the nonlinear Burgers operator proposed by Wang and Oberai,[23] which is defined as

$$\tau(u_N) = \left[3\pi\nu^2\left(\frac{4}{h^2}\right)^2 + \frac{4}{h^2}\|u_N\|^2\right]^{-\frac{1}{2}}, \tag{21}$$

where $h = \pi/N$ can be considered as an effective grid size and $\|u_N\|$ is the L^2-norm of u_N. This approximation basically aims to bound the effect of the nonlinear operator in Fourier space. Now, we can compute the subscales in terms of the resolvable scales from (20) and use the resulting expression in the equation for the latter.