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Variational multi-scale finite element approximation of the compressible Navier-Stokes equations

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Abstract

Purpose – The purpose of this paper is to apply the variational multi-scale framework to the finite element approximation of the compressible Navier-Stokes equations written in conservation form. Even though this formulation is relatively well known, some particular features that have been applied with great success in other flow problems are incorporated.

Design/methodology/approach – The orthogonal subgrid scales, the non-linear tracking of these subscales, and their time evolution are applied. Moreover, a systematic way to design the matrix of algorithmic parameters from the perspective of a Fourier analysis is given, and the adjoint of the non-linear operator including the volumetric part of the convective term is defined. Because the subgrid stabilization method works in the streamline direction, an anisotropic shock capturing method that keeps the diffusion unaltered in the direction of the streamlines, but modifies the crosswind diffusion is implemented. The artificial shock capturing diffusivity is calculated by using the orthogonal projection onto the finite element space of the gradient of the solution, instead of the common residual definition. Temporal derivatives are integrated in an explicit fashion.

Findings – Subsonic and supersonic numerical experiments show that including the orthogonal, dynamic, and the non-linear subscales improve the accuracy of the compressible formulation. The non-linearity introduced by the anisotropic shock capturing method has less effect in the convergence behavior to the steady state.

Originality/value – A complete investigation of the stabilized formulation of the compressible problem is addressed.

Keywords Compressible flow, Explicit Runge-Kutta scheme, Shock capturing, Stabilized finite elements, Variational multi-scale (VMS)

Paper type Research paper

1. Introduction

The compressible Navier-Stokes equations are commonly used to model flow problems when compressibility effects become relevant. Some areas that require a compressible flow description are the aerodynamics and aeroacoustics fields, in which applications range from turbo-machinery design to speech therapy. The compressible Navier-Stokes equations consists of the conservation of mass, momentum and energy equations. Thermodynamical properties and constitutive relations of the fluid close the mathematical description. We restrict our problem to perfect fluids in the gas state, which we model using Newton's law for fluids together with the caloric equation, the ideal gas law, and Fourier's heat law. However, other compressible fluids can be modeled by the compressible Navier-Stokes equations using the appropriate constitutive definitions.

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In this work we are interested in the finite element method (FEM) approximation of compressible flow problems. In particular, we solve this set of equations using the conservative variables formulation, that is, defining density, momentum, and total energy as the problem unknowns. When these equations are approximated by the classical Galerkin approach, numerical instabilities may appear due to the hyperbolic nature of the equations. The first attempt to deal with this instability was the introduction of a stabilizing term into the Galerkin finite element formulation. Within this concept, the Streamline Upwind Petrov Galerkin (SUPG) method by Tezduyar and Hughes (1983) was the first method adopted for solving the compressible Navier-Stokes equations. In that formulation, the authors applied the stabilization methods previously developed for the convection-diffusion equation. The main idea was to optimally introduce numerical diffusion along the streamlines using a stabilization term that contained a matrix of algorithmic parameters, a certain operator applied to the test function, and the residual of the differential equation (e.g. Brooks and Hughes, 1981). An important contribution of this pioneering work was the inclusion of the largest eigenvalue of the hyperbolic system into the matrix of algorithmic parameters. More recently, Polner (2005), Billaud *et al.* (2010), and Sevilla *et al.* (2013) applied the SUPG stabilization into their compressible flow formulations. Some later stabilizations were formulated based on the Galerkin Least Squares method. In all of these works (Shakib, 1989; Hauke and Hughes, 1998; Hauke *et al.*, 2005), the authors transformed the conservative variables into entropy variables using Jacobian transformation matrices.

The multi-scale concept was first included in the compressible Navier-Stokes formulations to account for the effect of the unresolved scales as a turbulence model instead of as a stabilization method. Within this branch, Koobus and Farhat (2004) proposed a mixed formulation that approximated the solution separating a priori the resolved and unresolved turbulent scales. They included the effect of unresolved scales into the solution using a Reynolds stress tensor. The unresolved scales were calculated with a finite volume cell-agglomeration projector. Van Der Bos *et al.* (2007) contrasted that initial work, specifically in the definition of the unresolved scales by means of the cell-agglomeration projector. They proposed instead a Fourier-Spectral projector which could be implemented with a discontinuous Galerkin method. Levasseur *et al.* (2006) and Dahmen *et al.* (2011) implemented the turbulence modeling concept and presented turbulence energy spectra results for turbulent compressible flows.

The stabilization methods proposed in this work are based on the variational multi-scale framework introduced by Hughes (1995) for the scalar convection-diffusion-reaction problem. The method splits the unknowns of the problem into a coarse-scale that belongs to the finite element space and a subgrid scale or subscale, which is the remainder. The original problem is subdivided into two separated problems, one for the finite element scale, and another for the subscales. To avoid increasing the number of degrees of freedom of the problem, an approximation over the subscales is done in terms of the resolved scales. In order to properly do this approximation, we propose to study the behavior of the compressible problem from the perspective of a Fourier analysis, instead of the previous approximations (Rispoli and Saavedra, 2006; Marras *et al.*, 2013). Hence, we aim to give a systematic way to design the approximation over the subscales, that is, to define the matrix of algorithmic parameters for the compressible case.

The main idea of the variational multi-scale stabilization formulation is to include the effect of the unresolved scales in order to stabilize the finite element equations. The variational multi-scale stabilized finite element formulation was introduced to solve the compressible Navier-Stokes equations by Rispoli and Saavedra (2006). They extended satisfactorily the formulation previously done for the incompressible Navier-Stokes equations by Rispoli *et al.* (2007). In that work, they demonstrated the matrix of algorithmic parameters definition for a one-dimensional advective-diffusive-reactive problem on quadratic elements. The contribution of the subscales into the resolved scales was done by integrating by parts the finite element scale equation. More recently, Marras *et al.* (2013) applied the variational multi-scale method in order to stabilize the Euler equations, and demonstrated convergence of the method in a widespread range of stratified flows. They implemented a linear explicit Euler time integration scheme. Even though the variational multi-scale formulation of the compressible problem has already been worked by the last mentioned authors, we incorporate some particular features that we have applied in other flow problems (e.g. Codina *et al.*, 2007; Baiges and Codina, 2010). Since there are different ways to define the subscales, three different approaches are solved in this work. The first possibility is the definition of the space of the subscales, which can be either the space of finite element residuals, or the space orthogonal to the finite element space (Codina, 2000, 2011). The second possibility is the inclusion of the transient term of the subscales equation (Codina *et al.*, 2007). The third possibility is the inclusion of the subscales in all the non-linear terms of the problem (Codina, 2001). In addition to these definitions, we take into account the volumetric part of the convective term in the compressible non-linear operator adjoint, which has not been proposed before in the stabilized formulations.

Furthermore, at supersonic regime some localized instabilities may arise from sharp gradients in the solution, which are inherent to the physics of the problem. Hence, the stabilized formulation requires to be complemented with a local shock capturing term, in order to yield stability and convergence in the entire domain. The first approach was the residual-based shock capturing techniques that controls oscillations in a non-linear fashion depending on the solution. This type of shock capturing operator was introduced by Hughes and Mallet (1986) and later by Le Beau and Tezduyar (1991) into the SUPG compressible flow formulation. Mittal and Tezduyar (1998), Mittal (1998), Hughes *et al.* (2010), and Kotteda and Mittal (2014) continued developing shock capturing stabilized formulations. In contrast with previous formulations (e.g. Tezduyar and Senga, 2006; Sevilla *et al.*, 2013; Woopen *et al.*, 2014) we aim to introduce the numerical diffusion in a "physical manner" for the compressible problem. That is, we modify the diffusion of the momentum and energy equations, but avoid introducing artificial diffusion into the mass equation. In this topic, we also propose to use the orthogonal projection onto the finite element space of the gradient of the unknown, instead of the common residual definition. Because the subgrid stabilization method works in the streamline direction, we implement an anisotropic shock capturing method that keeps the diffusion unaltered in the direction of the streamlines, but modifies the crosswind diffusion. The scope of this proposed shock capturing technique is evaluated with some numerical examples.

Solving the finite element system of equations in an implicit fashion can be computationally very expensive. Our various applications of the compressible formulation, some of which require the solution of small spatial and temporal scales (such as turbulent flows and aeroacoustics problems), have prompted us to implement explicit time integration schemes, which are reasonably in line with these needs. Therefore, we use an explicit time integration scheme in order to integrate the temporal derivatives, more precisely, we implement the family of explicit Runge-Kutta (RK) methods. Moreover, we calculate the time integration method for the temporal derivative of the dynamic subscales exploiting the explicit scheme adopted.

We perform numerical tests of the formulations presented. The first section of tests corresponds to the study of the formulations in the subsonic range, where there is no need for local stabilization and it is possible to compare the global stabilized formulations. We implement the three-dimensional lid-driven cavity and the flow past a cylinder problems, both in the subsonic range, and benchmark our solutions to those published by other authors. We use the cylinder results obtained with linear triangular elements to compare the convergence of the proposed methods. As another type of test, we make a comparison between the shock capturing methods (proposed in this paper) using the supersonic reflected shock problem. Finally, we present the solution for the compressible viscid supersonic flow past a cylinder including the global and local stabilization methods. This example closes the evaluation of the formulation.

This paper is organized as follows. In Section 2 we present the compressible Navier-Stokes problem. Section 3 contains the numerical approximation using the variational multi-scale finite element formulation of the compressible Navier-Stokes equations. We also present the demonstration of the matrix of stabilization parameters, and the distinct approaches to solve the subscales equation. The shock capturing methods and the explicit time integration scheme are also discussed in Section 3. In Section 4 the numerical examples are presented, and the proposed stabilization is discussed. Finally, in Section 5 conclusions are stated.

2. The compressible Navier-Stokes problem

2.1 Initial and boundary value problem

The problem we consider consists in the Navier-Stokes equations posed in a time interval $(0,T)$ and in a domain $\Omega\subset\mathbb{R}^d$, being d the number of space dimensions ($d=2$ or 3). Let $t\in(0,T)$ be a given time instant in the temporal domain, and $\boldsymbol{x}\in\Omega$ a given point in the spatial domain. Let Γ be the boundary of the domain Ω, and $\boldsymbol{n}$ the geometric unit outward normal vector on Γ. We split Γ into two sets: the Dirichlet boundary denoted as Γ_g, and the Neumman boundary denoted as Γ_n. Considering a compressible, Newtonian and viscous fluid, the governing equations are the conservation of mass, momentum, and energy written in conservation form:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}\left(\rho u_j u_i+p\delta_{ij}-\tau_{ji}\right)=\rho f_i, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}u_i u_i\right)\right)+\frac{\partial}{\partial x_j}\left(\rho u_j\left(\mathrm{h}+\frac{1}{2}u_i u_i\right)-u_i\tau_{ij}+q_j\right)=\rho f_i u_i+\rho r, \tag{3}$$