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The Fuzzy Project Scheduling Problem with Minimal Generalized Precedence Relations

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Abstract: *In scheduling, estimations are affected by the imprecision of limited information on future events, and the reduction in the number and level of detail of activities. Overlapping of processes and activities requires the study of their continuity, along with analysis of the risks associated with imprecision. In this line, this article proposes a fuzzy heuristic model for the Project Scheduling Problem with flows and minimal feeding, time and work Generalized Precedence Relations with a realistic approach to overlapping, in which the continuity of processes and activities is allowed in a discretionary way. This fuzzy algorithm handles the balance of process flows, and computes the optimal fragmentation of tasks, avoiding the interruption of the critical path and reverse criticality. The goodness of this approach is tested on several problems found in the literature; furthermore, an example of a 15-story building was used to compare the better performance of the algorithm implemented in Visual Basic for Applications (Excel) over that same example input in Primavera© P6 Professional V8.2.0, using five different scenarios.*

1 INTRODUCTION

Construction projects are developed under the constraints of scope, time, and budget. To achieve the time target, including interim deadlines, construction schedulers generally use a group of tools known as the critical path method, scheduling through a hierarchy of three layers from low to high levels of detail (Nicholas and Steyn, 2008). This cascade structure implies that for short-term planning, the growth in the level of detail increases the number of activities with simple interdependences between them; however, long-term planning implies that, on the one hand, the number of activities is reduced but, on the other hand, complex interdependences with overlapping and decisions on continuity of the activities are generated (Arditi and Bentotage, 1996). The proposal presented

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later in the article helps schedulers dealing with this issue.

The problem of overlapping and splitting the activities was studied for first time by Crandall (1973), who considered that disallowing the splitting of activities was an excessive relaxation of the real problem; he proposed a novel algorithm with discretional fragmentation. After Crandall, several authors (Wiest, 1981; Valls et al., 1996; Moder et al., 1983) have proposed different improvements to Crandall's algorithm but, to the best of the authors' knowledge, the problem of the overlapping and optimal fragmentation of activities is not totally solved yet. Furthermore, some of the activities of construction projects are grouped in processes (Hejducki, 2004) that cannot be addressed in the traditional way: the decision is not focused on the fragmentation of the activities but on the continuous execution (flow) between them throughout the process.

Additionally, long-term planning involves unavailable or incomplete information, which requires rough estimations in the forecast of project parameters, producing a high risk of failure (Nicholas and Steyn, 2008; Ballesteros-Pérez et al., 2013; Herroelen and Leus, 2005). To solve the problem of risk management in projects, statistical approaches have been proposed such as the Program Evaluation Review Technique (Malcolm et al., 1959), or the Monte Carlo Simulation Models based on random distributions (Alarcón et al., 2011).

Several authors argue that when the nature of the risk is associated not with the presence of random variables but with the epistemic uncertainty and the imprecision of the estimates produced by limited, incomplete, uncertain, or approximate information (Bonnal et al., 2004; Herroelen and Leus, 2005; Fougères and Ostrosi, 2013; Jahani et al., 2014), the use of the Theory of Fuzzy Sets is appropriate (Zadeh, 1965; Dell'Orco and Mellano, 2013; Zeng et al., 2014). Lootsma (1989) states that the Theory of Fuzzy Sets is closer to reality and simpler to use than stochastic models, facilitating the implementation of values that are not precisely known, but that can be limited within certain bounds of membership or fuzziness (Castro-Lacouture et al., 2009). Some estimations are true (or false) only to some extent, and computers may not process them, thus, fuzzy logic can be applied to extend computer capacities (Bai et al., 2014). In natural linguistic terms, when there is not sufficient information for a deterministic estimation or a statistical measurement, experts use their own judgment and experience with the available project information, with expressions such as "approximately" or "around" a minimum and a maximum value, in other words, "more-or-less" (Haque Khan and Akhtar Hasin, 2012).

Summarizing, available mathematical models for practitioners in construction scheduling and planning do not consider the actual complex, inherently imprecise and uncertain conditions of projects (Castro-Lacouture et al., 2009; Yan and Ma, 2013), challenging decisions on discretional splitting of activities and continuity of processes.

Therefore, to partially fill this gap, the authors propose a heuristic approach to the Project Scheduling Problem with Generalized Precedence Relationships applying the Theory of Fuzzy Sets, which allows the optimal splitting of activities and considers the processes flow. This way, this research contributes to the body of knowledge of construction project scheduling in two facets: (a) using fuzzy sets theory; and (b) implementing the splitting of activities and flow of processes. This approach takes into account the sometimes unavoidable interruption of activities at the construction site, improving other previously proposed algorithms, as well as the commercial software, and providing robust results. Furthermore, the use of fuzzy logic provides a friendlier environment for the practitioner than stochastics approaches.

To introduce this proposal properly, the following section provides a literature review of the state-of-knowledge of the Project Scheduling Problem with Generalized Precedence Relationships (GPSP from now on). Section 3 details the proposed model for the fuzzy Project Scheduling Problem with Generalized Precedence Relationships (fuzzy-GPSP hereafter) with flows and minimal generalized relationships for production planning in construction projects. In Section 4, sound examples found in literature are solved in different ways to check the goodness and versatility of the fuzzy-GPSP. An example of application is displayed in Section 5 to prove that the model can be rigorously implemented to assist practitioners; the fuzzy-GPSP is compared to Primavera© P6 Professional V8.2.0 in diverse scenarios presenting the differences and performance metrics. Finally, conclusions and limitations of the research are drawn.

2 LITERATURE REVIEW

The classical scheduling methods of Activity-On-Arrow graphs, such as the Critical Path Method (Kelley and Walker, 1959) and the Program Evaluation Review Technique (Malcolm et al., 1959), were developed simultaneously at the end of the 1950s by two separate organizations (the DuPont and Remington Rand and the U.S. Navy with Polaris missile project, respectively). They introduced the concept of precedence in a finish-to-start relationship ($FS_{ij}(z)$) in which activity j (successor) cannot start until activity i (predecessor) is totally finished. The times of each activity are computed by applying a forward-backward algorithm in a topologically ordered graph. Times obtained in the forward pass (ascending order) are known as the earliest starting time

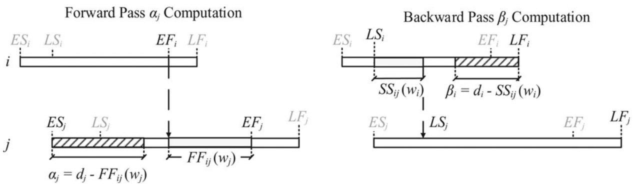

Fig. 1. Computing the fragments α_j and β_i in the Crandall (1973) and Valls et al. (1996) algorithms.

(ES_j) and the earliest finishing time (EF_j) of the activities, establishing the duration of the project (makespan) as the earliest finishing time of the last activity. When the algorithm is applied in descending order (backward pass), the latest starting (LS_j) time and the latest finishing (LF_j) time of the activities are obtained.

In addition to the classical finish-to-start relationship ($FS_{ij}(z)$) of the Activity-On-Arrow (Fondahl, 1961), more Generalized Precedence Relations (GPRs) applying Activity-On-Node graphs were proposed by Kerbosch and Schell (1975), IBM (1968), and Crandall (1973). The first authors (Kerbosch and Schell, 1975) proposed the so-called Extended METRA Potential Method, developed in France in 1958 (Roy, 1962), introducing for the first time the notion of "percentage relation for the begin-begin (start-to-start) relation." IBM (1968) and Crandall (1973) developed the Precedence Diagramming Method with GPRs as currently known. The new GPRs for the Precedence Diagramming Method are the start-to-start ($SS_{ij}(z)$), the finish-to-finish ($FF_{ij}(z)$), and the start-to-finish ($SF_{ij}(z)$) relationships.

Both methods, the Extended METRA Potential Method and the Precedence Diagramming Method, seem to be similar but are conceptually different. The Extended METRA Potential Method only computes the earliest starting and finishing times, considering activities as "no splitting allowed." Crandall (1973) considered that disallowing the splitting of activities is an excessive relaxation of the real problem, and presented a complete heuristic algorithm to compute the times for the activities and the minimum duration of the project. The algorithm is capable of recognizing the segments (α and β) (see Figure 1) belonging to the same activity and the decision of splitting the activities is discretionary for practitioners.

Crandall's algorithm was improved by Moder et al. (1983), including the start-to-finish relationship. More recently, Valls et al. (1996) analyzed Crandall's algorithm and the splitting criteria failures, proposing a new computation for the splitting parameters, as well as a new and more realistic treatment of the start-to-finish relationship; they considered simultaneously two different values for z, concerning its predecessor and successor, respectively. Another algorithm was proposed by Hajdu (1997), which involved relaxing the splitting criteria without considering the α and β segments.

For the "no splitting allowed" problem, the GPRs can be relaxed into the more familiar finish-to-start relations (Elmaghraby and Kamburowski, 1992) (standardized form), or into minimum start-to-start precedence relations (Bartusch et al., 1988; De Reyck and Herroelen, 1998, 1999). If more than one relationship exists between i and j, only the most restrictive must be considered. Unfortunately, in some cases, these relaxations provide infeasible solutions to the problem.

The Precedence Diagramming Method with GPRs presents anomalous effects that are counterintuitive about the consequences of lengthening or shortening a job (Wiest, 1981; Herbert and Deckro, 2011; Valls and Lino, 2001), changing the concept of the critical path itself. This anomalous effect, called "reverse criticality," is produced when a critical path passes through an activity from finish to start. Then, the activity's effect on the critical path is "perverse" (Crandall, 1973), that is, lengthening the activity shortens the critical path and shortening the activity lengthens the path; consequently, such a result is called a reverse criticality. Reverse criticality occurs because the z-value of the relationships is usually a function of the intensity in the execution of the activities; this aspect is not covered in the traditional formulation (Herbert and Deckro, 2011).

Another approach considers the use of artificial neural networks to optimize the scheduling problem, taking the neural dynamics model of Adeli and Park (1995) for structural optimization as its point of departure. Adeli and Karim (1997, 2001), Karim and Adeli (1999a, b), Adeli and Wu (1998), and Senouci and Adeli (2001) considered precedence relationships, repetitive and non-repetitive activities, work continuity, multiple crews, as well as the effect of changing job conditions on the performance of a crew in their scheduling models. All these models were based on the time-cost trade-off, seeking minimizing cost as well as time optimization.

Other approaches consider the intensity of the activities using the "feeding precedence" constraints (Kis et al., 2004; Kis, 2005, 2006), based on the model developed by Leachman et al. (1990) with overlapping execution of activities. The Leachman dependences were classified by Bianco and Caramia (2009, 2011, 2012) as *%Completed-to-start* (%CS) and additional feeding precedences are the *Start-to-%Completed* (S%C), *Finish-to-%Completed* (F%C), and *%Completed-to-Finish* (%CF). To solve the partial overlap and fragmentation of activities, Kis (2006) proposed the manual splitting of activities, introducing appropriate precedence constraints.

More recently, new approaches were proposed to face the problem of overlapped activities: Beeline Diagramming Method (Kim, 2012), Design Structure

Matrix (Srour et al., 2013), Collaborative Product Development Process (Zhang et al., 2013), and Concurrency Scheduling (Lim et al., 2014). The Beeline Diagramming Method represents the overlapping relationship of two activities with an arrow that indicates the direction of workflow, from any point of the predecessor to any point of the successor; it allows multiple linkage relationships between two activities, providing more realistic schedules in a hierarchy schedule (Kim, 2012). The Design Structure Matrix model, based on the work of Krishnan et al. (1997), finds the shortest possible (overlapped) design schedule by processing the dependence information gathered in the exchange among design activities. Collaborative Product Development process deals with local scheduling behavior of designer agents and resource conflicts resolution behavior of the manager agent allowing task interruption and rescheduling at any time of change. Concurrency-based scheduling keeps makespan and cost down by overlapping the predecessors and the successors without assigning additional resources, characterizing the activities by two attributes: evolution as the production rate of a predecessor activity, and sensitivity as the probability of rework´s occurrence in the successor when a change occurs in the predecessor (Lim et al., 2014).

The problem of the Repetitive Scheduling Method, that is repetitive activities organized in processes, has been deeply studied since O'Brien (1969) proposed the Line of Balance for projects of multi-story buildings, houses, and highways or pipelines (Damci et al., 2013). This method is widely accepted under different names as the Vertical Production Method (O'Brien, 1975), Linear Scheduling Method (Barrie and Paulson, 1978; Adeli and Karim, 1997; Ammar, 2013), Time-Space Scheduling Method (Stradal and Cacha, 1982), or Repetitive Scheduling Method (Harris and Ioannou, 1988; Maravas and Pantouvakis, 2011a); currently, it is more well-known as Location-Based Scheduling (Seppänen et al., 2014).

The first works applying fuzzy logic to project scheduling problem was by Prade (1979) and Chanas and Kamburowski (1981) applying fuzzy numbers with triangular membership functions based on optimistic, the most likely, and the pessimistic estimates of the respective activity durations. Chang et al. (1995) proposed a project planning based on fuzzy Delphi method, and Chen and Chang (2001) dealt with the problem of finding Multiple Possible Critical Paths using Fuzzy PERT.

One of the most controversial problems on fuzzy scheduling is problem of ranking (Brunelli and Mezei, 2013; Deng, 2014; Wang et al., 2006) and determining latest starting dates in a satisfactory manner. Dubois, Fargier, and Galva (2003) dealt with the problem of determining latest starting dates and slack times in a satisfactory manner (Chanas and Kamburowski, 1981; Rommelfanger, 1994; Dubois et al., 2003). Chen and Huang (2007) proposed a method to measure the fuzzy criticality in project network. Castro-Lacouture et al. (2009) applied fuzzy values to time, cost, and material restrictions using fuzzy mathematical models. Ponz-Tienda et al. (2012) proposed a fuzzy scheduling model integrating a novel interpretation of Earned Value Indexes. The latest proposal was by Chrysafis and Papadopoulos (2014) with an alternative approach to Fuzzy-PERT based on possibilistic estimate for the mean to avoid the problem of the backward recursion.

Approaches applying robust optimization models were proposed by Long and Ohsato (2006). Shi and Blomquist (2012) and Maravas and Pantouvakis (2011a, 2011b) proposed fuzzy durations for activities considering overlapping. The first authors (Shi and Blomquist, 2012) used the fuzzy Design Structure Matrix, indexing the time factor of information exchange along with the durations for each activity and their relationships in two different matrices: one for the predecessor activities and the other for the successor activities. The second authors (Maravas and Pantouvakis, 2011a) applied the fuzzy Repetitive Scheduling Method for the scheduling of repeating activities whose unit production rates were uncertain or imprecise. Finally, Ma et al. (2015) suggested a scenario-based proactive robustness optimization method, which aims to improve the soundness of construction project schedules that are developed using the critical chain project management method.

Throughout the analyzed literature, researchers have shown that the GPRs are needed to formulate "*real-life constraints*" (Schwindt, 2014), and that fragmentation and overlapping of processes and activities is very effective in improving the estimation of the project makespan for the Unconstrained Project Scheduling Problem (PSP) (Valls et al., 1996; Ponz-Tienda et al., 2011) and Constrained PSP (Quintanilla et al., 2012), as well as for fast-tracking complex construction projects (Srour et al., 2013) by setting up overlapping activities (Lim et al., 2014). Preemption as a solution for the fragmentation has been widely studied in manufacturing and services environments. Nonetheless, realistic approaches to the optimal splitting and overlapping of activities and processes in construction projects with imprecise values for durations and relationships is almost void (Lino et al., 2012) and still open for research (Schwindt, 2014).

Several scheduling techniques have been briefly presented in this section, but available mathematical models and decision support systems for practitioners hardly take into account the complex and ill-defined conditions of construction projects. Karim and Adeli (1999c) stated that "the limitations and shortcomings of the existing software systems used in practice are also recognized by the construction industry" (p. 381); however, the

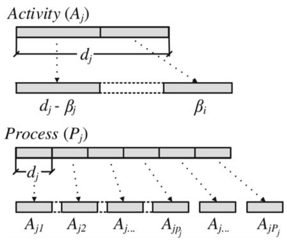

Fig. 2. Splitting criteria of processes and activities.

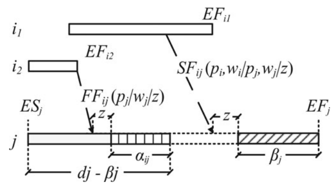

Fig. 3. Computing the fragment β_j.

traditional Critical Path Method (CPM) approach "is still being used despite its documented shortcomings, particularly for projects involving repeating tasks" (Karim and Adeli, 1999c, p. 380). Therefore, to partially fill this gap and make a contribution to the body of knowledge, the authors propose in the next section a heuristic approach to the fuzzy Project Scheduling Problem with Generalized Precedence Relationships allowing the optimal splitting of activities and considering the processes flow.

3 THE PROPOSED FUZZY-GPSP FOR PRODUCTION PLANNING IN CONSTRUCTION PROJECTS

For the purpose of this research, an activity is defined as a unique and basic work entity which consumes a certain quantity of resources with a constant intensity during its execution; an activity can be carried out by parts (Lopez and Roubellat, 2013). A process is defined as a set of *P* repetitive activities (called cycles) that must be executed strictly one after the other but not necessarily in a continuous way; they consume the same quantity of resources with a constant intensity during their execution.

The formulation of the proposed model for the Unconstrained GPSP is based on the following principles: (a) the activities of the project can be fragmented in a discretionary way according to the criteria of the practitioner; (b) under a functional point of view, a maximum of one interruption point per activity is considered (Valls et al., 1996); and (c) the activities belonging to the same process must be executed without interruption and with discretional continuity between them, according to the criteria of the practitioner (Figure 2). In real environments, activities and processes interact with each other, transmitting information in the form of "minimum production flows," which are necessary for the occurrence of future events in the successor activities and/or processes.

Before presenting the model, the traditional subtraction in the backward pass of the fuzzy-PSP needs to be explained to avoid further confusion. Computing the latest times (LS) of the activities is an important issue which can provide negative values for the left side of the obtained fuzzy number. This happens because the backward pass of the Pseudo-code 1 is strictly applied as with crisp values, without taking the precaution that computing the LS of the activities is calculated by solving an equation with the LS as the unknown term Equation (1):

$$LS_i + Dur_i = LF_i \rightarrow LS_i = LF_i - Dur_i \tag{1}$$

This fact does not have transcendental implications on scheduling with crisp values, but with fuzzy values the equation must be solved as a Minkowski´s subtraction (Buckley and Eslami, 2002; Chrysafis and Papadopoulos, 2014) of the supports of the fuzzy number Equation (2), guarantying that the obtained solution always culminates in a positive confidence interval (Gil-Aluja, 2004). If a fuzzy difference is applied, negative values can be obtained for the times of the activities, providing incorrect solutions:

$$\begin{aligned}
&\overline{LS}_i(x) \oplus \overline{Dur}_i(x) = \overline{LF}_i(x)\\
&(ls_{i1}, ls_{i2}, ls_{i3}) \oplus (d_{i1}, d_{i2}, d_{i3}) = (lf_{i1}, lf_{i2}, lf_{i3})\\
&\begin{cases} ls_{i1} = \min(ls_{i2}, lf_{i1} - d_{i1})\\ ls_{i2} = lf_{i2} - d_{i2}\\ ls_{i3} = \max(ls_{i2}, lf_{i3} - d_{i3}) \end{cases}
\end{aligned} \tag{2}$$

In the model introduced in this article, the criteria for computing the times of the activities is an evolution of Crandall's proposal (1973), including the Valls et al. (1996) start-to-finish relationship, with a different formulation for determining the splits of the activities and the Latest Starting (LS) times, which avoid unfeasible solutions providing optimal project makespan. The proposed algorithm is based on establishing the value of the fragment β_j in two phases. In the first phase, β_j is initialized as the minimum "work/feeding GPR" restriction that affects the finishing of the activity Equation (3). Once the times of the successors are computed, β_j is recomputed in a second phase (Figure 3), as the minimum feasible value that meets the constraints of the problem, applying Equation (3).

$$\beta_j = \min(FF_{ij}|SF_{ij}), \forall i \ sucessor \ of \ j \tag{3}$$

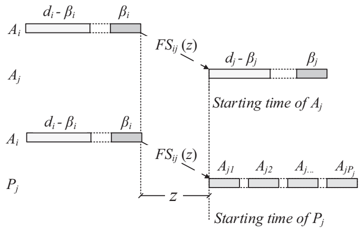

Fig. 4. Finish-to-start ($\overline{FS}_{ij}(z)$) relationship between activities and activities-processes.

$$EF_i \leq ES_j - d_j - \beta_j \rightarrow \begin{cases} True \rightarrow \alpha_{ij} = d_j - EF_i - ES_j - \beta_j \\ False \rightarrow \alpha_{ij} = 0 \end{cases}$$
$$\beta_j = \max(\beta_j, p_j \cdot d_j + w_j - \alpha_{ij}) \tag{4}$$

The possible relationships between activities and processes are explained in the following paragraphs, with the criteria and pseudo-code for determining the times.

The finish-to-start $\overline{FS}_{ij}(z)$ precedence relationship between activities (or activities-processes) (Figure 4) represents the minimum number of z "time units" that must elapse between the completion of the predecessor activity, A_i, and the start of the follower activity, A_j (or activity A_{jp} of the process P_j).

The pseudo-code to compute the times for the fuzzy finish-to-start $\overline{FS}_{ij}(\overline{z}(x))$ precedence relationship is presented in Pseudo-code 1:

Pseudo-code 1: fuzzy FS relationship

Early Times

CASE $predecessor(i) \rightarrow successor(j)$ **OF**

Case $activity(i) \rightarrow activity(j)$:

$[ES_j]^\alpha = \overline{\max}\left([ES_j]^\alpha, [EF_i]^\alpha \oplus [z]^\alpha\right)$

$[EF_j]^\alpha = \overline{\max}\left([EF_j]^\alpha, [ES_j]^\alpha \oplus [d_j]^\alpha\right)$

Case $activity(i) \rightarrow process(j)$:

$[ES_{j1}]^\alpha = \overline{\max}\left([ES_{j1}]^\alpha, [EF_i]^\alpha \oplus [z]^\alpha\right)$

$[EF_{j1}]^\alpha = [ES_{j1}]^\alpha \oplus [d_j]^\alpha$

CALL *FrWComputeProccessesTimes*$(j,2)$

END CASE

Latest Times

CASE $predecessor(i) \rightarrow successor(j)$ **OF**

Case $activity(i) - activity(j)$:

$[LF_i]^\alpha = \overline{\min}\left([LF_i]^\alpha, [LS_j]^\alpha - [z]^\alpha\right)$

$[LS_i]^\alpha = \overline{\min}\left([LS_i]^\alpha, [LF_i]^\alpha - [d_i]^\alpha\right)$

Case $activity(i) \rightarrow process(j)$:

CALL *BckWComputeProccessesTimes*$(j,1)$

$[LF_i]^\alpha = \overline{\min}\left([LF_i]^\alpha, [LS_{j1}]^\alpha - [z]^\alpha\right)$

$[LS_i]^\alpha = \overline{\min}\left([LS_i]^\alpha, [LF_i]^\alpha - [d_i]^\alpha\right)$

END CASE

Note: the arithmetic operator – is the traditional (not fuzzy) subtraction operator (Eq. 2)

The feeding/work and time start-to-start $SS_{ij}(p_i|w_i|z)$ precedence relationship between activities (or activities-processes) (Figure 5) represents the minimum number of $p_i \cdot d_i$ or/and w_i "work units" ($0 \leq p_i < 1$, $w_i < d_i$) required on the predecessor activity, A_i, prior to the start of the successor activity, A_j (or process P_j), with an additional lag of z "time units."

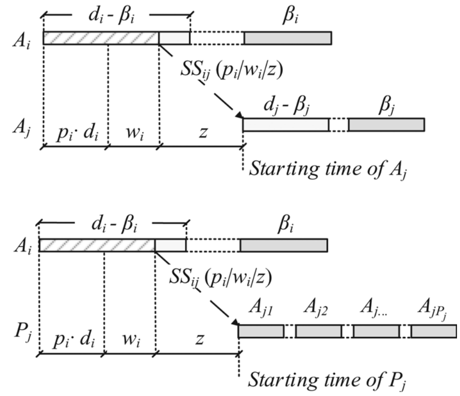

Fig. 5. Feeding/work and time start-to-start ($SS_{ij}(p_i|w_i|z)$).

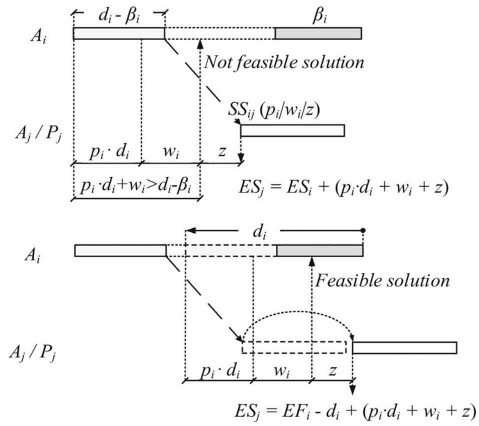

Fig. 6. Feasible/not feasible solution for the $SS_{ij}(p_i|w_i|z)$.

In the forward pass for computing the earliest starting (ES_j) times of the successor activity or process, it is necessary to verify the feasibility of the solution when the predecessor activity is to be split. When $p_i \cdot d_i + w_i > d_i - \beta_i$, the minimum production flow required for starting the successor is not met, so it is necessary to delay the starting time of the successor (A_j/P_j) to a feasible position from the finishing time of the predecessor (A_i), as can be seen in the bottom part of Figure 6.

The Pseudo-code 2 is presented to compute the times for the fuzzy start-to-start $SS_{ij}(\overline{p}_i(x)|\overline{w}_i(x)|\overline{z}(x))$ precedence relationship, including a new computation for the Latest Starting times based on a parameter k_{ij}:

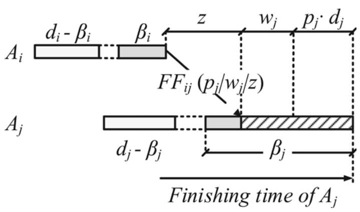

Fig. 7. Feeding/work and time finish-to-finish ($FF_{ij}(p_j|w_j|z)$).

Pseudo-code 2: fuzzy SS precedence relationship
IF $[d_i]^\alpha - [\beta_i]^\alpha < [p_i]^\alpha \otimes [d_i]^\alpha \oplus [w_i]^\alpha$ **THEN**
$[\boldsymbol{k}_{ij}]^\alpha = [EF_i]^\alpha - [ES_i]^\alpha - [d_i]^\alpha \oplus \left([p_i]^\alpha \otimes [d_i]^\alpha \oplus [w_i]^\alpha\right)$
ELSE IF : $[\boldsymbol{k}_{ij}]^\alpha = [p_i]^\alpha \otimes [d_i]^\alpha \oplus [w_i]^\alpha$
END IF
Early Times
CASE *predecessor*$(i) \rightarrow$ *successor*(j) **OF**
Case *activity*$(i) \rightarrow$ *activity*(j):
$[ES_j]^\alpha = \overline{\max}\left([ES_j]^\alpha, [ES_i]^\alpha \oplus [\boldsymbol{k}_{ij}]^\alpha \oplus [z]^\alpha\right)$
$[EF_j]^\alpha = \overline{\max}\left([EF_j]^\alpha, [ES_j]^\alpha \oplus [d_j]^\alpha\right)$
Case *activity*$(i) \rightarrow$ *process*(j)
$[ES_{j1}]^\alpha = \overline{\max}\left([ES_{j1}]^\alpha, [ES_i]^\alpha \oplus [\boldsymbol{k}_{ij}]^\alpha \oplus [z]^\alpha\right)$
$[EF_{j1}]^\alpha = [ES_{j1}]^\alpha \oplus [d_j]^\alpha$
CALL *FrWComputeProccessesTimes*$(j,2)$
CALL *FrWComputeProccessesTimes*$(j,2)$
END CASE
Latest Times
CASE *predecessor*$(i) \rightarrow$ *successor*(j) **OF**
Case *activity*$(i) \rightarrow$ *activity*(j)
$[LS_i]^\alpha = \overline{\min}\left(\overline{LS}_i[\alpha], \overline{LS}_j[\alpha] - \overline{\boldsymbol{k}}_{ij}[\boldsymbol{\alpha}] - \overline{z}[\alpha]\right)$
Case *activity*$(i) \rightarrow$ *process*(j)
CALL *BckWComputeProccessesTimes*(j, P_j)
$[LS_i]^\alpha = \overline{\min}\left([LS_i]^\alpha, [LS_{j1}]^\alpha - [\boldsymbol{k}_{ij}]^\alpha - [z]^\alpha\right)$
END CASE
Note: the arithmetic operator – is the traditional (not fuzzy) subtraction operator (Eq. 2)

The feeding/work and time finish-to-finish precedence relationship ($FF_{ij}(p_j|w_j|z)$) between activities (Figure 7) represents the minimum number of $p_j \cdot d_j$ or/and w_j "work units" ($0 \le p_j < 1, w_j < d_j$) required on the follower activity, A_j, after the completion of its predecessor, A_i, with an additional lag of z "time units."

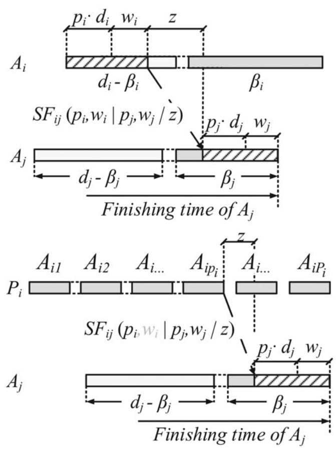

Fig. 8. Feeding/work and time start-to-finish ($SF_{ij}(p_i, w_i|p_j, w_j|z)$).

The Pseudo-code 3 is presented to compute the times for the fuzzy finish-to-finish $FF_{ij}(\overline{p}_j(x)|\overline{w}_j(x)|\overline{z}(x))$ precedence relationship:

Pseudo-code 3 fuzzy FF precedence relationship
Early Times
$[aux]^\alpha = [EF_i]^\alpha \oplus [p_j]^\alpha \otimes [d_j]^\alpha \oplus [w_j]^\alpha \oplus [z]^\alpha$
$[EF_j]^\alpha = \overline{\max}\left([EF_j]^\alpha, [\mathrm{aux}]^\alpha\right)$
IF $[EF_j]^\alpha \le [ES_j]^\alpha - [d_j]^\alpha - [\beta_j]^\alpha$ **THEN**
$[\alpha_{ij}]^\alpha = [d_j]^\alpha - [EF_i]^\alpha - [ES_j]^\alpha - [\beta_j]^\alpha$
ELSE IF : $[\alpha_{ij}]^\alpha = \overline{0}$
END IF
$[\beta_j]^\alpha = \overline{\max}\left([\beta_j]^\alpha, [p_j]^\alpha \otimes [d_j]^\alpha - [w_j]^\alpha - [\alpha_{ij}]^\alpha\right)$
IF *j is continuous* **THEN** $[ES_i]^\alpha = [EF_i]^\alpha - [d_j]^\alpha$
Latest Times
$[LF_i]^\alpha = \overline{\min}\left([LF_i]^\alpha, [LF_j]^\alpha - [p_j]^\alpha \otimes [d_j]^\alpha - [w_j]^\alpha - [z]^\alpha\right)$
$[LS_i]^\alpha = \overline{\mathbf{min}}\left([LS_i]^\alpha, [LF_i]^\alpha - [d_i]^\alpha\right)$
IF *j is not due to be split* **THEN** $[LF_j]^\alpha = [LS_i]^\alpha \oplus [d_j]^\alpha$
Note: the arithmetic operator – is the traditional (not fuzzy) subtraction operator (Eq. 2)

The feeding/work and time start-to-finish precedence relationship ($SF_{ij}(p_i, w_i|p_j, w_j|z)$) between activities (Figure 8) represents the minimum number of $p_j \cdot d_j$ or/and w_j "work units" ($0 \le p_j < 1$, $w_j < d_j$) required on the follower activity, A_j, after the minimum number of $p_i \cdot d_i$ or/and w_i "work units" ($0 \le p_i < 1$, $w_i < d_i$) on the predecessor activity, A_i, has been completed, with an additional lag of z "time units."

This approach for the start-to-finish relationship ($SF_{ij}(p_i, w_i|p_j, w_j|z)$) is based on the formulation proposed by Valls et al. (1996), and suits a greater diversity of conditions more closely to reality. The traditional formulation ($SF_{ij}(z)$) only considers the units required between the finish of the activity and the start of the predecessor; this fact implies the paradox, in production terms, that a follower activity can finish without a "real start" of the predecessor.

The procedure to compute the times for the fuzzy finish-to-finish $SF_{ij}\,(\overline{p}_i(x),\ \overline{w}_i(x)|\ \overline{p}_j(x),\ \overline{w}_j(x)|\ \overline{z}(x))$ precedence relationship is presented in the Pseudo-code 4.

Pseudo-code 4: fuzzy SF precedence relationship

Early Times

CASE $predecessor(i) \rightarrow successor(j)$ **OF**

Case $activity(i) \rightarrow activity(j)$:

Case $activity(i) \rightarrow activity(j)$

  IF $[d_i]^\alpha - [\beta_i]^\alpha < [p_i]^\alpha \otimes [d_i]^\alpha \oplus [w_i]^\alpha$ **THEN**

  $$[\boldsymbol{k_{ij}}]^\alpha = [EF_i]^\alpha - [ES_i]^\alpha - [d_i]^\alpha \oplus \left([p_i]^\alpha \otimes [d_i]^\alpha \oplus [w_i]^\alpha\right) \oplus \oplus \left([p_j]^\alpha \otimes [d_j]^\alpha \oplus [w_j]^\alpha\right)$$

  ELSE IF :

  $$[\boldsymbol{k_{ij}}]^\alpha = \left([p_i]^\alpha \otimes [d_i]^\alpha \oplus [w_i]^\alpha\right) \oplus \left([p_j]^\alpha \otimes [d_j]^\alpha \oplus [w_j]^\alpha\right)$$

  END IF

  $$[EF_j]^\alpha = \overline{\max}\left([EF_j]^\alpha, [ES_i]^\alpha \oplus [\boldsymbol{k_{ij}}]^\alpha \oplus [z]^\alpha\right)$$

Case $process(i) \rightarrow activity(j)$

  $$[EF_j]^\alpha = \overline{\max}\left([EF_j]^\alpha, [EF_{ip_i}]^\alpha \oplus [p_j]^\alpha \otimes [d_j]^\alpha \oplus [w_j]^\alpha \oplus [z]^\alpha\right)$$

END CASE

IF $[EF_j]^\alpha \leq [ES_j]^\alpha - [d_j]^\alpha - [\beta_j]^\alpha$ **THEN**

  $$[\alpha_{ij}]^\alpha = [d_j]^\alpha - [EF_i]^\alpha - [ES_j]^\alpha - [\beta_j]^\alpha$$

ELSEIF $[\alpha_{ij}]^\alpha = \overline{0}$

END IF

$$[\beta_j]^\alpha = \overline{\max}\left([\beta_j]^\alpha, [p_j]^\alpha \otimes [d_j]^\alpha - [w_j]^\alpha - [\alpha_{ij}]^\alpha\right)$$

IF *j must be continuous* ***THEN*** $[d_j]^\alpha = [EF_i]^\alpha - [ES_j]^\alpha$

Latest Times

CASE $predecessor(i) \rightarrow successor(j)$ **OF**

Case $activity(i) \rightarrow activity(j)$

  $$[LF_i]^\alpha = \overline{\min}\left([LF_i]^\alpha, [LF_j]^\alpha - [\boldsymbol{k_{ij}}]^\alpha - [z]^\alpha\right)$$

  $$[LS_i]^\alpha = \overline{\min}\left([LS_i]^\alpha, [LF_i]^\alpha - [d_i]^\alpha\right)$$

Case $process(i) \rightarrow activity(j)$

  $$[LF_{ip_i}]^\alpha = \overline{\min}\left([LF_{ip_i}]^\alpha, [LF_j]^\alpha - [z]^\alpha\right)$$

  $$[LS_{ip_i}]^\alpha = \overline{\min}\left([LS_{ip_i}]^\alpha, [LF_{ip_i}]^\alpha - [d_i]^\alpha\right)$$

  CALL $BckWComputeProccessesTimes(i,\ p_i)$

END CASE

Note: the arithmetic operator – is the traditional (not fuzzy) subtraction operator (Eq. 2)

The flow and time $(Fl_{ij}(p_i|p_j|z))$ relationship between processes or activity-processes (Figure 9) represents the minimum number of p_i cycles (or $p_i{\cdot}d_i$) required on the predecessor process, P_i (or activity A_i),

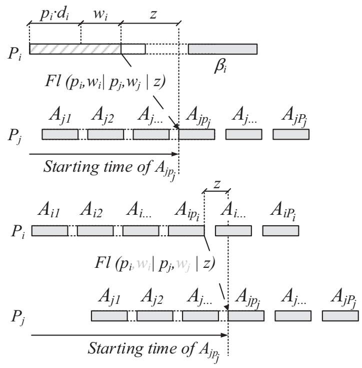

Fig. 9. Flow and time $(Fl_{ij}(p_i,\, w_i|p_j,\, w_j|z))$.

prior to the starting of process P_j, with an additional lag of z "time units." This formulation for the flow and time relationship is an adaptation of the Valls et al. (1996) proposal of the start-to-finish precedence relationship between activities.

The algorithm to compute the times for the fuzzy flow $(Fl_{ij}(\overline{p}_i(x),\, \overline{w}_i(x)|\overline{p}_j(x),\, \overline{w}_j(x)|\overline{z}(x)))$ relationship is presented in the Pseudo-code 5.

Pseudo-code 5 fuzzy flow precedence relationship

Early Times

CASE $predecessor(i) \rightarrow successor(j)$ **OF**

Case $activity(i) \rightarrow process(j)$:

  IF $[d_i]^\alpha - [\beta_i]^\alpha < [p_i]^\alpha \cdot [d_i]^\alpha + [w_i]^\alpha$ **THEN**

  $$[\boldsymbol{k_{ij}}]^\alpha = [EF_i]^\alpha - [d_i]^\alpha \oplus \left([p_i]^\alpha \otimes [d_i]^\alpha \oplus [w_i]^\alpha\right)$$

  ELSE IF : $\overline{\boldsymbol{k}}_{\boldsymbol{ij}}[\boldsymbol{\alpha}] = \overline{p}_i[\alpha] \otimes \overline{d}_i[\alpha] \oplus \overline{w}_i[\alpha]$

  END IF

  $$[ES_{jp_j}]^\alpha = \overline{\max}\left([ES_{jp_j}]^\alpha, [ES_i]^\alpha \oplus [\boldsymbol{k_{ij}}]^\alpha \oplus [z]^\alpha\right)$$

  $$[EF_{jp_j}]^\alpha = [ES_{jp_j}]^\alpha \oplus [d_j]^\alpha$$

Case $process(i) \rightarrow process(j)$:

  $$[ES_{jp_j}]^\alpha = \overline{\max}\left([ES_{jp_j}]^\alpha, [EF_{ip_i}]^\alpha \oplus [z]^\alpha\right)$$

  $$[EF_{jp_j}]^\alpha = \overline{\max}\left([EF_{jp_j}]^\alpha, [ES_{jp_j}]^\alpha \oplus [d_j]^\alpha\right)$$

END CASE

CALL $FrWComputeProccessesTimes(j,\ p_{j+1})$

Latest Times

CALL : $BckWComputeProccessesTimes(j,\ P_j)$

CASE $predecessor(i) \rightarrow successor(j)$ **OF** :

Case $activity(i) \rightarrow process(j)$

  $$[LS_i]^\alpha = \overline{\min}\left([LS_i]^\alpha, [LS_{jp_j}]^\alpha - [\boldsymbol{k_{ij}}]^\alpha - [z]^\alpha\right)$$

Case $process(i) \rightarrow process(j)$

  $$[LF_{ip_i}]^\alpha = \overline{\mathbf{min}}\left([LF_{ip_i}]^\alpha, [LS_{jp_j}]^\alpha - [z]^\alpha\right)$$

  $$[LS_{ip_i}]^\alpha = \overline{\min}\left([LS_{ip_i}]^\alpha, [LF_{ip_i}]^\alpha - [d_i]^\alpha\right)$$

CALL $BckWComputeProccessesTimes(i,\ p_i)$

END CASE

Note: the arithmetic operator – is the traditional (not fuzzy) subtraction operator (Eq. 2)

The algorithm for computing the times of the processes in the forward (*FrWComputeProcessesTimes* (j, p_j)) and backward (*BckWComputeProcessesTimes* (j,p_i)) pass is shown in Pseudo-code 6.

Pseudo-code 6 Forward/Backward pass of processes

$FrWComputeProcc\_Times(j, p)$

FOR $k = p$ **to** P_j **step** 1
  $\left[ES_{jk}\right]^{\alpha} = \overline{\min}\left(\left[ES_{jk}\right]^{\alpha}, \left[EF_{jk-1}\right]^{\alpha}\right)$
  $\left[EF_{jk}\right]^{\alpha} = \overline{\min}\left(\left[EF_{jk}\right]^{\alpha}, \left[ES_{jk}\right]^{\alpha} \oplus \left[d_j\right]^{\alpha}\right)$
END FOR
IF *j is due to be continouss* ***THEN***
$\left[ES_{j1}\right]^{\alpha} = \overline{\min}\left(\left[ES_{j1}\right]^{\alpha}, \left[EF_{jP}\right]^{\alpha} - P_j \otimes \left[d_j\right]^{\alpha}\right)$
$\left[EF_{j1}\right]^{\alpha} = \overline{\min}\left(\left[EF_{j1}\right]^{\alpha}, \left[ES_{j1}\right]^{\alpha} \oplus \left[d_j\right]^{\alpha}\right)$
FOR $k = 2$ **to** P_j **step** 1
  $\left[ES_{jk}\right]^{\alpha} = \overline{\min}\left(\left[ES_{jk}\right]^{\alpha}, \left[EF_{jk-1}\right]^{\alpha}\right)$
  $\left[EF_{jk}\right]^{\alpha} = \overline{\min}\left(\left[EF_{jk}\right]^{\alpha}, \left[ES_{jk}\right]^{\alpha} \oplus \left[d_j\right]^{\alpha}\right)$
END FOR
  END IF

$BckWComputeProcc\_Times(j, p)$

FOR $k = p$ **to** 1 **step** 1
$\left[LF_{jk}\right]^{\alpha} = \overline{\min}\left(\left[LF_{jk}\right]^{\alpha}, \left[LS_{jk+1}\right]^{\alpha}\right)$
$\left[LS_{jk}\right]^{\alpha} = \overline{\min}\left(\left[LS_{jk}\right]^{\alpha}, \left[LF_{jk}\right]^{\alpha} - \left[d_j\right]^{\alpha}\right)$
END FOR
IF *j is due to be continous* ***THEN***
FOR $k = 2$ **to** P_j **step** 1
  $\left[ES_{jk}\right]\alpha = \overline{\min}\left(\left[ES_{jk}\right]^{\alpha}, \left[EF_{jk-1}\right]^{\alpha}\right)$
  $\left[EF_{jk}\right]^{\alpha} = \overline{\min}\left(\left[EF_{jk}\right]^{\alpha}, \left[ES_{jk}\right]^{\alpha} \oplus \left[d_j\right]^{\alpha}\right)$
END FOR
  END IF

Note: the arithmetic operator – is the traditional (not fuzzy) subtraction operator (Eq. 2)

The compiled heuristic algorithm for the fuzzy-GPSP, presented in the Pseudo-code 7, implements Pseudo-codes 1 to 6, and summarizes the procedure. The previous step consists in establishing the value for the project starting time ($\overline{ProjectStart}(\alpha)$), the number of activities/processes, and the interval for the alpha cuts (*alphainterval*), which must be a real number between zero and one ($0\le$ *alphainterval* ≤ 1). The proposed algorithm is simpler than the existing ones and corrects the errors that other approaches have, especially by computing the latest times.

Pseudo-code 7 Compiled algorithm for the fuzzy-GPSP

SET $\overline{ProjectStart}(\alpha) = (0,0,0)$
SET *NumberofActivities*
SET *alphainterval* | $0 \le$ *alphainterval* ≤ 1

Phase 0 : Preliminary $\overline{\boldsymbol{\beta}}(\ )$ **computations**

FOR *alpha* $= 0$ **to** 1 **step** *alphainterval*
FOR $i = 1$ **to** *NumberofActivities* **step** 1
IF *predecessor = activity*
AND relationship is (FF_{ij} **OR** SF_{ij}) **THEN**
$[\beta_i]^{\alpha} = \max([p_j]^{\alpha} \otimes [d_j]^{\alpha} + [w_j]^{\alpha})$
END IF
END FOR
END FOR

Phase 1 : Forward - Pass computations

$\left[ES_1\right]^{\alpha} = \left[ProjectStart\right]^{\alpha}$
FOR *alpha* $= 0$ **to** 1 **step** *alphainterval*
FOR $j = 1$ **to** *NumberofActivities* **step** 1
FOR $i = 1$ **to** $j - 1$ **step** 1
CASE *relationship* **OF**
Case Finish-to-Start;
  Call Pseudo - code 1: *finish · start _ Fw*
Case Start-to-Start;
  Call Pseudo - code 2: *start · start _ Fw*
Case Finish-to-Finish;
  Call Pseudo - code 3: *finish · finish _ Fw*
Case Start-to-Finish;
  Call Pseudo - code 4: *start · finish _ Fw*
Case Flow relationship;
  Call Pseudo - code 5: *Flow _ Fw*
  Call Pseudo - code 6: *Fw*
END CASE
END FOR
END FOR
END FOR

Phase 2 : Backward - Pass computations

FOR *alpha* $= 0$ **to** 1 **step** *alphainterval*
FOR $i =$ *NumberofActivities* **to** 1 **step** -1
FOR $j = i + 1$ **to** *NumberofActivities* **step** 1
CASE *relationship* **OF**
Case Finish-to-Start;
  Call Pseudo - code 1 : *start · start _ Bw*
Case Start-to-Start;
Call Pseudo - code 2 : *start · start _ Bw*
Case Finish-to-Finish;
  Call Pseudo - code 3 : *finish · finish _ Bw*
Case Start-to-Finish;
  Call Pseudo - code 4 : *start · finish _ Bw*
Case Flow relationship;
  Call Pseudo - code 5 : *Flow _ Bw*
  Call Pseudo - code 6: *Bw*
END CASE
END FOR
END FOR
END FOR

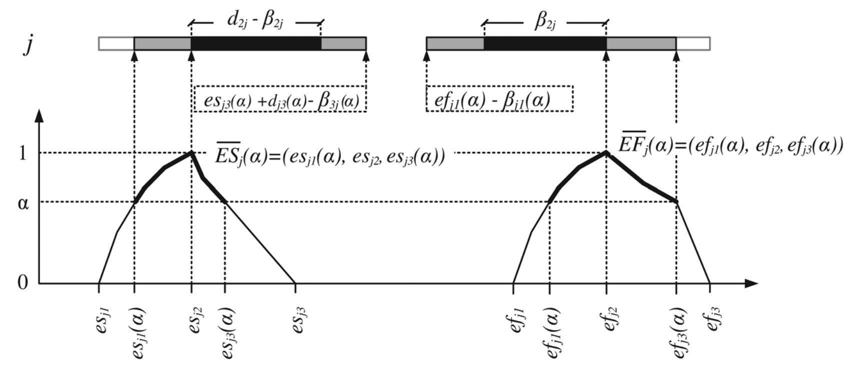

Fig. 10. Interpretation of fuzzy times to build the temporal diagram for a specific alpha cut.

The previous fuzzy values for the earliest starting times $[ES_j]^\alpha = (es_{j1}(\alpha), es_{j2}, es_{j3}(\alpha))$ and the earliest finishing times $[EF_j]^\alpha = (ef_{j1}(\alpha)_j, ef_{j2}, ef_{j3}(\alpha))$ of the activities must be interpreted as depicted in Figure 10, based on the values of $[\beta_j]^\alpha = (\beta_{j1}(\alpha)_j, \beta_{j2}, \beta_{j3}(\alpha))$. To build the temporal diagram on its latest times, the earliest starting times $[ES_j]^\alpha$ and the earliest finishing times $[EF_j]^\alpha$, they must be changed by the latest starting time $[LS_j]^\alpha = (ls_{j1}(\alpha), ls_{j2}, ls_{j3}(\alpha))$ and the latest finishing times $[LF_j]^\alpha = (lf_{j1}(\alpha), lf_{j2}, lf_{j3}(\alpha))$, respectively.

The Total Float $(\overline{TF}(x))$ of the activities must be computed in the traditional way (Equation (5)) as the fuzzy difference between the Latest Finishing $(\overline{LF}(x))$ time, the Early Starting $(\overline{ES}(x))$ time and the duration $(\overline{d}(x))$ of the activities. Additionally, the floats of each one of the fragments of the activities can be computed obtaining the starting $(\overline{stF}(x))$ (Equation (6)) and finishing $(\overline{fsF}(x))$ floats (Equation (7)) of the activities.

$$\overline{TF}(x) = \overline{LF}(x) \ominus \overline{ES}(x) \ominus \overline{d}(x) \quad (5)$$

$$\overline{stF}(x) = \overline{LS}(x) \ominus \overline{ES}(x) \quad (6)$$

$$\overline{fsF}(x) = \overline{LF}(x) \ominus \overline{EF}(x) \quad (7)$$

The fuzziness in the floats enhances the notion of criticality itself, making it possible to rank the activities in different degrees of criticality through the Critical Index (CI_j) and the Critical Value (CV_j) which established the criticism degree and the risk of criticality respectively.

In this way, an activity belongs totally to the set of critical activities (membership equal to 1) if the vertex of the fuzzy float is zero $(f_{j2} = 0)$. When the vertex of the fuzzy float is a positive value $(f_{j2}>0)$ and the left support is a negative value $(f_{j1} \leq 0)$, the activity is only critical for certain degrees of vagueness from zero to the Critical Index (CI_j). The Critical Index (CI_j) is the value of the alpha cut for which all the values of the float's subset are greater than zero (Equation 7 and Figure 11). Values

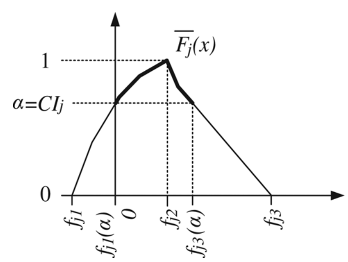

Fig. 11. Critical Index (CI_j).

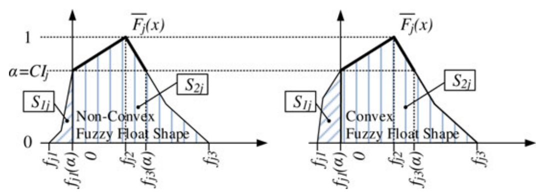

Fig. 12. Critical Value (CV_j).

near to one represent more criticality than values near to zero.

The same value for the Critical Index (CI_j) can be obtained for infinite shapes with identical vertex and supports of $\overline{F}(x)$ depending on their convexity (Figure 12), but convex shapes present more risk of criticality than non-convex shapes for the same Critical Index (CI_j). The Critical Value (CV_j) establishes the risk of criticality, and is computed by applying Equation (9) as the relation of areas (left side (S_{1j}) between right side (S_{2j}) from zero of the fuzzy float $(\overline{F}(x))$ multiplied by the Critical Index (CI_j). However, values near to one represent more risk of criticality than values near to zero.

$$\forall \overline{F_j}(x) = (f_{j1} \leq 0, f_{j2} > 0, f_{j3} > 0) \exists CI_j | \quad (8)$$

$$[F_j]^{CI_j} = [f_{j1}(CI_j), f_{j3}(CI_j)] = \{CI_j \in \mathbb{R} : f_{j1}(CI_j) = 0\}$$

$$CV_j = (S_{1j}/S_{2j}) \cdot CI_j \quad (9)$$

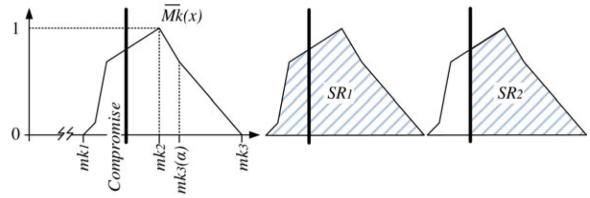

Fig. 13. Risk Index (*RI*) of fuzzy makespan.

In a similar way, the risk of accomplishing the project makespan (Figure 13) can be computed over the compromise date, establishing the Risk Index (*RI*) as the relation between the area to the right side of the compromise date and the total area of the fuzzy makespan by applying Equation (10).

$$RI = SR_2 / SR_1 \tag{10}$$

4 COMPARISON TO PREVIOUS RELEVANT PROBLEMS

The proposed fuzzy-GPSP model needs numerical experimentation and comparison to test its reliability. However, libraries cited in the literature for benchmarking (e.g., Demeulemeester and Herroelen, 2002; Valls et al., 1996) are not available for this comparison due to the fact that researchers did not explicitly provide them; moreover, the Project Scheduling Problem Library (Kolisch and Sprecher, 1996) was developed for finish-to-start relationships and does not consider the GPRs. Therefore, the authors follow other approaches. For the numerical experimentation some relevant problems identified in the literature are solved in different ways. These relevant problems were proposed by Crandall (1973), Valls et al. (1996), Maravas and Pantouvakis (2011a), Kim (2012), and Shi and Blomquist (2012). They are computed again using the proposed fuzzy-GPSP and the results are compared to the original solution; Table 1 summarizes how each of them considers feeding, work and time GPRs and flow, as well as their main limitations. They are briefly discussed next.

The Crandall (1973) problem was reproduced and computed with the proposed fuzzy-GPSP algorithm considering "splitting allowed" for all the activities, durations as crisp values, and all the relationships between the activities as "Work GPRs." The problem was solved with the fuzzy-GPSP algorithm, obtaining the same times for the activities. Due to the simplicity of the problem, the goodness of Crandall's proposal was not conclusive. In fact, some failures were detected by Valls et al. (1996), especially on the splitting criteria and the computation of the latest starting (*LS*) times, which can produce discontinuities in the critical path when criticality is at the beginning and involves the β fragment of an activity.

The problem used by Valls et al. (1996) in their research was solved with the following criteria: all the durations are taken as crisp values and the relationships as "Work GPRs." The project makespan and the early times obtained are the same as the original results provided by the authors. However, Valls et al.'s algorithm (1996) is difficult to interpret and implement, presenting some relatively ambiguous aspects such as the criterion for calculating the latest starting (*LS*) times of the activities. These values are not included in their work and are not entirely consistent with the conclusions obtained by applying their formulation.

The Maravas and Pantouvakis (2011a) six-unit repetitive project, presented by Harris and Ioannou (1988), for testing the fuzzy Repetitive Scheduling Method has been reproduced considering: (1) activity *A* as three continuous processes $A_{1,2}$, $A_{3,4}$, and $A_{5,6}$; (2) activity *B* as one splittable process of six activities ($B_{1\text{-}6}$); (3) activity *C* as one continuous process of four activities ($C_{1\text{-}4}$) and an additional activity C_6; and (3) activities *E* and *F* as continuous processes ($E_{1\text{-}6}$ / $F_{1\text{-}6}$). The values obtained when applying fuzzy-GPSP to the problem are the same as those obtained by the authors for the vertex (crisp values), and seem to be the same for the supports of the fuzzy times that are not explicitly stated. The only difference is in activity *B* because of the discretional interruption between units 3 and 4 ("work break to accommodate the delivery").

The problem used by Kim (2012) for the Beeline Diagramming Method has been solved considering all the activities as "no splitting allowed," durations as crisp values, and transforming all precedence relationships (*Np-Ns*) into start-to-start GPRs ($SS(p = 0|w = Np\text{-}Ns|0)$) when necessary. The results obtained are totally coincident in time and criticality with those presented by the author.

The Shi and Blomquist (2012) problem is the most recent proposal found in the literature, with all the durations as fuzzy values and the relationships as "feeding GPRs." For a proper transformation, the overlapping established by the time factor is transformed into start-to-start relationships Equation (10), because they represent the best fit to the nature of the model proposed by the authors:

$$SS_{ij}\left(\overline{p}_i(x)\,|\,\overline{w}_i(x)\,|\,\overline{z}(x)\right)| \begin{cases} \overline{p}_i(x) = 0 \\ \overline{w}_i(x) = \overline{SS}_{ij} \otimes \left(\overline{B}_{ij}(x) \otimes \overline{d}_i(x) - \left(\overline{C}_{ij}(x) \otimes \overline{d}_j(x)\right)\right) \\ \overline{z}(x) = 0 \end{cases} \tag{11}$$

Table 1
Algorithms: characteristics, limitations, and comparison

Algorithm	*Feeding* *SS, FF, SF*	*Work* *SS, FF, SF*	*Time* *FS, SS, FF, SF*	*Flow* *Fl*	*Limitations*
Crandall (1973)	x, x, x	√, √, *x*	x, x, x, x	x	Splitting criteria *LS* computation
Valls et al. (1996)	x, x, x	√, √, √	x, x, x, x	x	Complex computation *LS* computation
Maravas and Pantouvakis (2011a)	x, x, x	x, x, x	√, x, x, x	√	Only time FS and flow relationships
Kim (2012)	x, x, x	√, x, x	√, x, x, x	√	No splitting allowed Only FS/SS relationships
Shi and Blomquist (2012)	√, x, x	x, x, x	x, x, x, x	x	Only one GPR Fuzzy arithmetic
Fuzzy-GPSP	√, √, √	√, √, √	√, √, √, √	√	—

Note: √ indicates supported relationship, x indicates non-supported relationship.

The differences observed with the Shi and Blomquist (2012) problem are due to the computation of the overlapping established by the times factor. In Equation (10) the "$-$" sign is a not fuzzy operation and this interpretation provides more trustworthy values for the supports of the fuzzy times (all values are positives for the times of the activities).

The algorithms displayed in Table 1 deal with the problem of overlapping in activities and processes. These algorithms are key contributions to the state-of-knowledge of simultaneity and fragmentation. As it can be inferred from Table 1, the proposed fuzzy-GPSP outperforms the others in these five facets: (a) it considers all the feeding GPRs (only the algorithm proposed by Shi and Blomquist (2012) takes into account the SS), avoiding reverse criticality; (b) it allows the use of the four work and time GPRs; (c) it improves Crandall (1973) and Valls et al. (1996) approaches with a new computation of the fragmentation avoiding the interruption of the critical path; (d) it includes the balance of process flows (only embraced previously by Maravas and Pantouvakis (2011a) and Kim (2012)); and (e) it presents an innovative formulation of fuzzy values for durations and relationships (even though a fuzzy approach was already proposed by Shi and Blomquist (2012), this one is more complete and robust).

5 EXAMPLE OF APPLICATION

As an example of implementation of the algorithm, a building of 15 floors is selected. The first 3 floors are under ground and the remaining 12 are above ground. To ensure the constructability of the foundations, the starting of "excavation phase 2" (a split activity) requires at least 14 days from the starting of the "water drainage" to guarantee that the water table is controlled. Additionally, the "water drainage" must work without interruption until at least 7 floors of the structure are completely finished to offset the water pressure with its weight. This fact implies that the duration of the "water drainage" is unknown and depends on the times of the followers and especially of the structure. The concrete of the foundations is scheduled in three phases, and overlapped with the reinforcement bars. The structure is a process of 15 activities overlapped with the processes of masonry, facades and basements with an additional lag of 12 days for removal of formwork to ensure the proper hardening of the concrete. A total of 18 activities/processes which summarize 78 activities and sub-processes are considered, contemplating the widest possible set of conditions.

The fuzzy values for the durations of the processes/activities, the number of activities by process and the five scenarios analyzed are displayed in Table 2. The nature of the relationships between activities and/or processes is shown in Table 3.

The example of application of the proposed fuzzy-GPSP algorithm has been implemented in Visual Basic for Applications (Excel 2013) (Ponz-Tienda et al., 2013) for a proper comprehension of the versatility and goodness of the proposal (Figure 14). This application can be downloaded from http://goo.gl/VZR0Ta, and allows

Table 2
Example of application: description of activities, continuity and duration

Code	Description process/activity	Continuity scenarios # 1	# 2	# 3	# 4	# 5	No. of activities	Duration $\overline{d}(d_1, d_2, d_3)$
1	Previous works	Yes	Yes	Yes	Yes	Yes	1	d(3,5,6)
2	Excavations 0.0/-1.0	Yes	Yes	Yes	Yes	Yes	1	d(12,12,14)
3	Water drainage	Yes	Yes	Yes	Yes	Yes	1	Dependent
4	Diaphragm-wall	Yes	Yes	Yes	Yes	Yes	1	d(45,45,50)
5	Excavations	Yes	Yes	Yes	Yes	Yes	1	d(28,30,35)
6	Rebars for foundation works	Yes	Yes	Yes	Yes	Yes	1	d(15,18,20)
7	Concrete foundation Ph.1	Yes	Yes	Yes	Yes	Yes	1	d(1,1,3)
8	Concrete foundation Ph.2	Yes	Yes	Yes	Yes	Yes	1	d(1,1,3)
9	Concrete foundation Ph.3	Yes	Yes	Yes	Yes	Yes	1	d(1,1,3)
10	Structure	Yes	Yes	Yes	Yes	Yes	15	d(11,12,14)
11	Decks	No	No	No	No	Yes	1	d(10,15,25)
12	Masonry works	No	No	No	Yes	Yes	12	d(3,4,5)
13	Facades	No	Yes	No	No	Yes	12	d(6,7,8)
14	Basements	No	No	No	No	Yes	3	d(25,30,35)
15	Paving works	No	No	Yes	No	Yes	12	d(5,6,8)
16	Office works	No	No	No	No	Yes	12	d(10,11,12)
17	Reworks and finishing	No	No	No	No	Yes	1	d(25,35,45)
18	Delivery/reception	Yes	Yes	Yes	Yes	Yes	1	d(1,1,1)

Table 3
Relationships between processes and activities

Code	Description	Predecessor # 1	# 2	# 3	# 4
1	Previous works	–			
2	Excavations 0.0/-1.0	$FS_{1\text{-}2}$(z)			
3	Water drainage	$FS_{2\text{-}3}$(z)	$SF_{11\text{-}3}$(p,w,z)		
4	Diaphragm-wall	$FS_{2\text{-}4}$(z)			
5	Excavations	$SS_{3\text{-}5}$(p,w,z)	$FS_{4\text{-}5}$(z)		
6	Rebars for foundation works	$FS_{5\text{-}6}$(z)			
7	Concrete foundation Ph.1	$SS_{6\text{-}7}$(p,w,z)			
8	Concrete foundation Ph.2	$SS_{6\text{-}8}$(p,w,z)	$FS_{7\text{-}8}$(z)		
9	Concrete foundation Ph.3	$FS_{6\text{-}9}$(z)	$FS_{8\text{-}9}$(z)		
10	Structure	$FS_{9\text{-}10}$(z)			
11	Decks	$FS_{10\text{-}11}$(z)			
12	Masonry works	$Fl_{10\text{-}12}(p_i,p_j,z)$			
13	Facades	$Fl_{10\text{-}13}(p_i,p_j,z)$			
14	Basements	$Fl_{10\text{-}14}(p_i,p_j,z)$			
15	Paving works	$Fl_{12\text{-}15}(p_i,p_j,z)$			
16	Office works	$Fl_{13\text{-}16}(p_i,p_j,z)$	$Fl_{15\text{-}16}(p_i,p_j,z)$		
17	Reworks and finishing	$Fl_{14\text{-}17}(p_i,p_j,z)$	$Fl_{16\text{-}17}(p_i,p_j,z)$	$FF_{14\text{-}17}$(p,w,z)	$FF_{16\text{-}17}$(p,w,z)
18	Delivery/reception	$FS_{17\text{-}18}$(z)			

anyone to modify the values for the fuzzy times, the type of relationships, the release date, and the continuity of activities and processes.

The app computes the fuzzy times, fuzzy floats, critical index, and critical value of activities and processes. It also calculates the risk of accomplishing the project makespan over the release date, and plots the fuzzy time for any of the activities by their *ES*, *EF*, *LS*, and *LF* (Figures 15 and 16). Additionally, it permits changes in the precision of the computations modifying the

alfa	0,00
α Interval	0,1
Discretize	Yes

Long Description	Short	#Activities	inf	nuc	sup	Continuity	ES(α) inf	ES(α) nuc	ES(α) sup	EF(α) inf	EF(α) nuc	EF(α) sup
Previous Works	Previous	1	3	5	6	Yes	0,0	0,0	0,0	3,0	5,0	6,0
Excavations 0.0/-1.0	Excav. Ph1	1	12	12	14	Yes	3,0	5,0	6,0	15,0	17,0	20,0
Water drainage	Drainage	1	165	183	220	Yes	15,0	17,0	20,0	184,0	200,0	236,0
Diaphragm-wall	Diaph.-wall	1	45	45	50	Yes	15,0	17,0	20,0	60,0	62,0	70,0
Excavations -1.0/-9.0	Excav. Ph2	1	28	30	35	Yes	60,0	62,0	70,0	88,0	92,0	105,0
Rebars for foundation	Reb. Found.	1	15	18	20	Yes	88,0	92,0	105,0	103,0	110,0	125,0
Concrete Foundation Ph1	Cte. Ph1	1	1	1	3	Yes	94,0	100,0	115,0	95,0	101,0	118,0
Concrete Foundation Ph2	Cte. Ph2	1	1	1	3	Yes	97,0	103,0	119,0	98,0	104,0	122,0
Concrete Foundation Ph3	Cte. Ph3	1	1	1	3	Yes	103,0	110,0	125,0	104,0	111,0	128,0
Structure	Structure	15	11	12	14	Yes	104,0	111,0	128,0	269,0	291,0	338,0
Decks	Decks	1	10	15	25	Yes	269,0	291,0	338,0	279,0	306,0	363,0
Masonry works	Masonry	12	3	4	5	No	158,0	171,0	202,0	282,0	307,0	361,0
Facades	Facades	12	6	7	8	No	180,0	195,0	230,0	297,0	324,0	380,0
Basements	Basements	3	25	30	35	No	158,0	171,0	202,0	233,0	261,0	307,0
Paving works	Paving	12	5	6	8	No	161,0	175,0	207,0	287,0	313,0	369,0
Offices Works	Offices	12	10	11	12	No	208,0	226,0	266,0	335,0	365,0	424,0
Reworks & Finishing work	Reworks	1	25	35	45	No	229,0	249,0	292,0	340,0	385,0	449,0
Delivery/reception	Makespan	1	1	1	1	Yes	340,0	385,0	449,0	341,0	386,0	450,0

Long Description	LS(α) inf	LS(α) nuc	LS(α) sup	LF(α) inf	LF(α) nuc	LF(α) sup	TF inf	TF nuc	TF sup	C Index	C Value
Previous Works	0,0	0,0	0,0	3,0	5,0	6,0	-3,0	0,0	3,0	1	1
Excavations 0.0/-1.0	3,0	5,0	6,0	15,0	17,0	20,0	-5,0	0,0	5,0	1	1
Water drainage	15,0	17,0	20,0	184,0	200,0	236,0	-56,0	0,0	56,0	1	0,78
Diaphragm-wall	15,0	17,0	20,0	60,0	62,0	70,0	-10,0	0,0	10,0	1	1
Excavations -1.0/-9.0	60,0	62,0	70,0	88,0	92,0	105,0	-17,0	0,0	17,0	1	1
Rebars for foundation	88,0	92,0	105,0	103,0	110,0	125,0	-22,0	0,0	22,0	1	1
Concrete Foundation Ph1	101,0	108,0	119,0	102,0	109,0	122,0	-16,0	8,0	27,0	0,7	0,25
Concrete Foundation Ph2	102,0	109,0	122,0	103,0	110,0	125,0	-19,0	6,0	27,0	0,8	0,39
Concrete Foundation Ph3	103,0	110,0	125,0	104,0	111,0	128,0	-24,0	0,0	24,0	1	1
Structure	104,0	111,0	128,0	269,0	291,0	338,0	127,0	168,0	223,0	0	0
Decks	275,0	295,0	339,0	285,0	310,0	364,0	-78,0	4,0	85,0	1	0,82
Masonry works	207,0	223,0	267,0	320,0	348,0	404,0	113,0	173,0	243,0	0	0
Facades	197,0	212,0	256,0	305,0	332,0	388,0	67,0	130,0	202,0	0	0
Basements	260,0	286,0	317,0	335,0	376,0	422,0	98,0	175,0	239,0	0	0
Paving works	210,0	227,0	272,0	325,0	354,0	412,0	110,0	173,0	246,0	0	0
Offices Works	215,0	233,0	280,0	335,0	365,0	424,0	57,0	128,0	206,0	0	0
Reworks & Finishing work	315,0	350,0	404,0	340,0	385,0	449,0	3,0	101,0	195,0	0	0
Delivery/reception	340,0	385,0	449,0	341,0	386,0	450,0	-109,0	0,0	109,0	1	1

Fig. 14. Implementation of the fuzzy-GPSP algorithm in Visual Basic for Applications (Excel 2013).

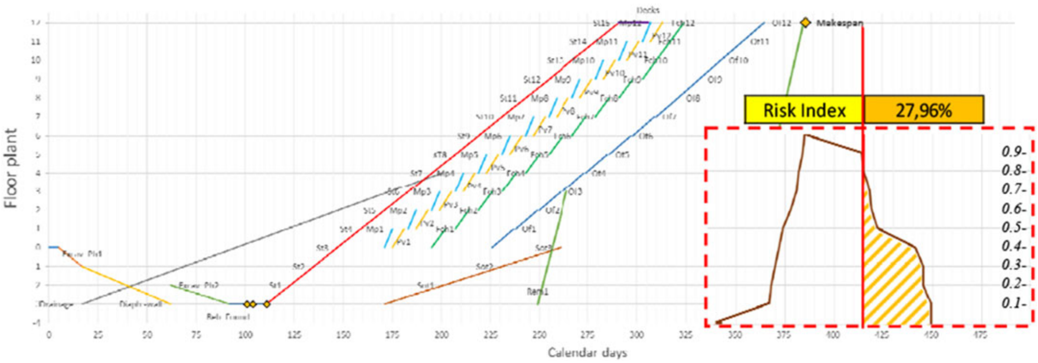

Fig. 15. Fuzzy makespan, risk index and space-time chart for scenario #1.

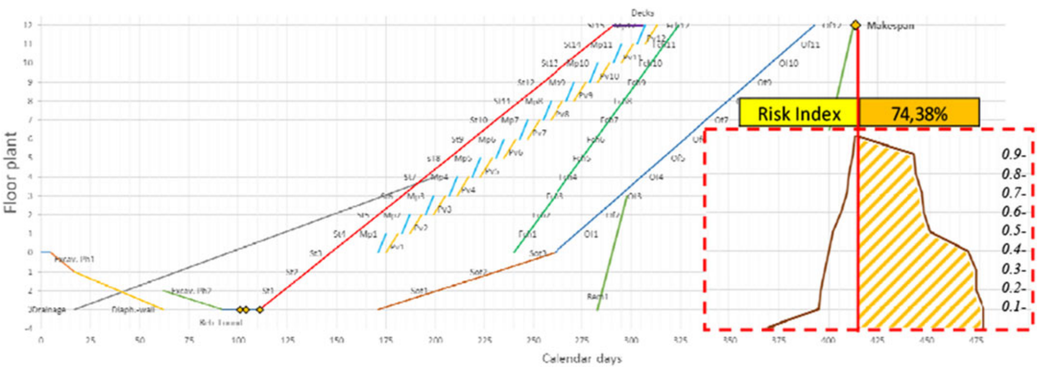

Fig. 16. Fuzzy makespan, risk index and space-time chart for scenario #2.

alphainterval, and the discretization of the fuzzy times in real days. The app has been partially implemented with C# to test the CPU time required to compute the fuzzy times and floats obtaining a mean of $6.67 \cdot 10^{-5}$ seconds for a problem with 35 activities and 10 alpha cuts using an Intel® Core™ i7-4770 processor at 3.40GHz and 8Gb (RAM).

The example of application included in the app has been solved for five different scenarios with the same compromise date (415 days) and values for durations and relationships. In scenario #1, activities 1 to 10 and 18 are considered as no splitting allowed, processes 12 to 16 as no continuous, and activities 11 and 17 as splitting allowed obtaining the fuzzy makespan represented in Figure 15 ($RI = 27.96\%$). In scenario #2, process 13 (facades) has been changed to continuous, increasing the *RI* of accomplishment up to 74.38% (Figure 16) given that the relation between the area to the right side of the compromise date (red line in Figures 15 and 16) and the total area is bigger in scenario #2 (Figure 16) than in scenario #1 (Figure 15). In scenarios #3 and #4, process 15 (paving works) and process 12 (masonry works) are considered continuous, respectively; for scenario #3, the fuzzy makespan is (357.0, 399.0, 450.0) and the *RI* is 43.83%; for scenario #4 the fuzzy makespan increases to (379.0, 421.0, 483.0) and the *RI* to 85.87%. Finally,

Table 4
Performance metrics of the example of application

	ES makespan						$\sum \overline{TF}_i$		
	$es_1(0)$	es_2	$es_3(0)$	*RI*	$\Sigma\ CI_i$	$\Sigma\ CV_i$	$\sum tf_1(0)$	$\sum tf_2$	$\sum tf_3(0)$
Scenario 1	340.0	385.0	449.0	27.96%	10.5	9.24	216.0	1066.0	1939.0
Scenario 2	368.0	413.0	477.0	74.38%	10.2	8.56	240.0	1112.0	1985.0
Scenario 3	357.0	399.0	450.0	43.83%	10.5	8.90	147.0	947.0	1729.0
Scenario 4	379.0	421.0	483.0	85.87%	11.5	9.53	125.0	980.0	1828.0
Scenario 5	379.0	421.0	483.0	85.87%	11.5	9.53	–120.0	749.0	1623.0

in scenario #5, all the activities and processes are considered continuous. The metrics obtained for the fuzzy makespan, Risk Index (*RI*), sum of the Critical Index ($\Sigma\ CI_i$), Critical Value ($\Sigma\ CV_i$), and fuzzy Total Float ($\sum \overline{TF}_i$) of all processes and activities are shown in Table 4.

In Table 4, it can be observed that the continuity of processes and activities usually increases the makespan of the project from (340.0, 385.0, 449.0) for scenario #1 to (379.0, 421.0, 483.0) in scenarios #4 and #5, and the risk of accomplishment from 27.96% (scenario #1) to 85.87% (scenarios #4 and #5). This risk of failure in the accomplishment of the project makespan is evidenced by comparing the critical metrics; for scenario #1, the sum of total floats ($\sum \overline{TF}_i$) is a positive fuzzy value (216.0, 1066.0, 1939.0) which provides a buffer of security along the project. In scenario # 5, ensuring the continuity for all the activities and processes, $\sum \overline{TF}_i$ is negative on the left side indicating greatest compliance risk.

To check the goodness of the proposal, the scenario #1 of the example of application has been scheduled and compared with Primavera© P6 Professional V8.2.0 (P6P). First (Solution A), the model has been solved considering the same activities/processes and the needed relationships for each algorithm that best considers the project restrictions. The project makespan provided by P6P is 409 days versus 385 for the optimal makespan with fuzzy-GPSP (Table 5); additionally, P6P presents 276 differences on scheduled times and 69 on float values of activities compared to the proposed fuzzy-GPSP. Later (Solution B), the model was scheduled with P6P without restrictions in the number of activities with the aim to obtain the same values provided by fuzzy-GPSP; for Solution B, P6P required 78 activities with 137 relationships versus 18 activities/processes and 25 relationships needed with the fuzzy-GPSP.

Then, to analyze the versatility of the fuzzy-GPSP, the number of operations needed in the transition from scenario #1 to #5 was computed (considering all the activities and processes continuous); the results are summarized in Table 6, comparing the obtained makespan and the number of differences in scheduled times and floats. P6P needed 60 operations versus the 7 operations used by fuzzy-GPSP. Furthermore, the solution obtained with P6P is not an optimal solution, presenting 48 differences in scheduled times and 33 in floats compared to the proposed fuzzy-GPSP.

The scenario #5 was selected as it provides the most unbiased analysis. The procedure used with P6P to guarantee the continuity of processes and activities is to establish the primary constraint of the activity status “As Late As Possible,” from the last activity to the first in topological order, consuming the free float of all the activities. With this procedure, the differences depend on the topological position in the graph, and they are reduced when all activities are considered in the analysis.

The fragmentation of construction activities and processes with many work intervals, restarts and interruptions is questionable from a practitioner’s point of view. Continuity is desirable to reduce direct cost, but overlapping and fragmentation reduces the risk of accomplishment and the project makespan. The proposed model does not consider the cost of disruption because, in the authors’ opinion, the benefits of improving the project makespan and reducing the risk of not accomplishment could be significantly more relevant than the cost of disruption. Nonetheless, it is discretional, and fairly straightforward, for schedulers to consider it in their decision-making.

The proposed fuzzy-GPSP is an important innovation on construction project scheduling, providing a robust and friendly decision support system, not only in its theoretical nature but in real life projects too. The limitations of the proposed model have to be analyzed more in-depth, although it has been tested with P6P as well as with some relevant problems identified in the literature that have been successfully solved, matching or improving their original solutions, despite the differences between them.

Table 5
Compared results for case 1 with Primavera© P6 Professional V8.2.0

							Differences	
		Model	*Activities*	*Relationships*	*Makespan*	*Optimal*	*Time*	*Floats*
Scenario 1	Solution A	Fuzzy-GPSP	18	25	385.0	Yes	–	–
		Primavera 6	18	31	409.0	No	276	69
	Solution B	Primavera 6	78	137	385.0	Yes	0	0

Table 6
Compared results for transition from scenario #1 to scenario #5

							Differences	
	Model	*Activities*	*Relationships*	*Makespan*	*Number of Operations*	*Optimal*	*Time*	*Floats*
Scenario 5	Fuzzy-GPSP	18	25	421.0	7	Yes	–	–
	Primavera 6	78	137	385.0	60	No	48	33

6 CONCLUSIONS

Scheduling of construction projects involves problems related to the overlapping of processes and activities, reverse criticality, fragmentation, continuity, and the use of unavailable or incomplete information. These facts produce a high risk of failed forecasts where a realistic approach in imprecise scenarios is not totally solved. The algorithms proposed by previous researchers only cope with some kinds of Generalized Precedence Relations and partially fail to provide satisfactory solutions to long-term scheduling problems.

Therefore, with the aim of helping to fill this gap, this article presents a heuristic approach to the Project Scheduling Problem with Precedence Relations applying the Theory of Fuzzy Sets, which allows the splitting of activities and considers the optimal processes flow. The compiled heuristic algorithm for the fuzzy-GPSP is presented in the Pseudo-code 7, being simpler than the existing ones and corrects the errors that other approaches have, especially by computing the latest times. It computes the fuzzy times, fuzzy floats, critical index, and critical value of activities and processes; it also calculates the risk of accomplishing the project makespan over the release date, plotting the fuzzy time for any of the activities by their ES, EF, LS, and LF. Additionally, it permits changes in the precision of the computations modifying the alphainterval, and the discretization of the fuzzy times in real days.

To test the performance of the model, it has been compared to previous algorithms proposed by other authors, discussing its capabilities. Furthermore, the model has been implemented in Visual Basic for Applications (Excel 2013) and applied to a building of 15 floors with a total of 18 activities/processes and 25 relationships, which summarizes 78 activities/subprocesses and 137 relationships, comparing the obtained schedules with Primavera© P6 Professional (V8.2.0) in five different scenarios and transitions between them.

This article contributes to the body of knowledge of construction project scheduling in several ways:

1. It comprises a complete state-of-knowledge of overlapping and splitting activities in the Project Scheduling Problem.
2. It presents a more realistic formulation of the fuzzy arithmetic for computing the latest starting times of the activities, which avoids the negative values.
3. It puts forward a fuzzy heuristic algorithm for the unconstrained case which improves and corrects previous contributions, computing unequivocally the fragmentation of activities.
4. It proposes a model for construction scheduling that takes into consideration all the feeding, work and time Generalized Precedence Relations, allowing the splitting of activities and continuity of processes in a discretionary way, as well as the balance of process flows.
5. The proposed model avoids the interruption of the critical path and the reverse criticality issue.

Schedulers and users can use the model to overcome some of the problems of the current algorithms, because these barely consider the complex and ill-defined conditions of construction projects. To begin with, the proposed algorithm handles all the feeding, work and time Generalized Precedence Relations and computes the fragmentation avoiding the interruption of the critical path. The model avoids reverse criticality by using feeding precedence relationships instead of the classical work days; this approach suits the natural thinking style of schedulers, who initially analyze and estimate the amounts, production flows and interdependences between activities. This anomaly implies that practitioners have to be alert if they use commercial software with Generalized Precedence Relations when adjusting the project duration by modifying the duration of the critical path or in the process of rescheduling the project. This can provide incorrect schedules, with errors difficult to detect and resolve, especially with a great number of activities. Furthermore, this algorithm includes the balance of process flows, letting the scheduler analyze the effects of overlapping and continuity of processes or activities over the makespan, as well as to deal with a single process instead of multiple different activities. Its implementation in a professional application can help practitioners to schedule real and complex projects in imprecise environments, modelling the project according to their needs and thinking style in an easy way, without having to adapt the real problem to the imposed relaxation of commercial software.

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NOTATIONS

The following concepts, symbols, and acronyms are used in this article:

Concepts, symbols, and acronyms	*Meaning*
$\overline{A}(x) = \{x, \mu(x) \vert x \in \Omega\}; \overline{A}(x) = (a_1, a_2, a_3)$	Fuzzy number of Ω (fuzzy set of real numbers)
$[a_1, a_3]$	Support of a fuzzy number (membership equals zero)
a_2	Vertex of a fuzzy number (membership equals one)
$[A]^\alpha = [a_1(\alpha), a_3(\alpha)]$	Alpha cut (alpha interval) of $\overline{A}(x)$
$[a_1(\alpha), a_3(\alpha)] = [a_1 + \alpha \cdot (a_2 - a_1), a_3 + \alpha \cdot (a_2 - a_3)]$	Alpha interval calculation
$[A]^\alpha \# [B]^\alpha = [C]^\alpha = [c_1(\alpha), c_3(\alpha)]$	Conceptual fuzzy arithmetic by intervals
$c_1(\alpha) = \mathbf{min}\begin{pmatrix} a_1(\alpha) \# b_1(\alpha), \\ a_1(\alpha) \# b_3(\alpha), \\ a_3(\alpha) \# b_1(\alpha), \\ a_3(\alpha) \# b_3(\alpha) \end{pmatrix}; \mathbf{c_3}(\alpha) = \mathbf{max}\begin{pmatrix} a_1(\alpha) \# b_1(\alpha), \\ a_1(\alpha) \# b_3(\alpha), \\ a_3(\alpha) \# b_1(\alpha), \\ a_3(\alpha) \# b_3(\alpha) \end{pmatrix}$	Fuzzy arithmetic by intervals
$\overline{C}(x) \geq \{\overline{A}(x), \overline{B}(x), \ldots, \overline{Z}(x)\}$ $[C]^\alpha = [c_1(\alpha), c_3(\alpha)]; \forall \alpha \vert 0 \leq \alpha \leq 1$ $\begin{cases} c_1(\alpha) = \mathbf{max}(\mathbf{a_1}(\alpha), \mathbf{b_1}(\alpha), \ldots, \mathbf{z_1}(\alpha)) \\ c_3(\alpha) = \mathbf{max}(\mathbf{a_3}(\alpha), \mathbf{b_3}(\alpha), \ldots, \mathbf{z_3}(\alpha)) \end{cases}$	Fuzzy number $\overline{C}(x)$ strictly greater than or equal to a given set of fuzzy numbers
$\overline{C}(x) \leq \{\overline{A}(x), \overline{B}(x), \ldots, \overline{Z}(x)\}$ $[C]^\alpha = [c_1(\alpha), c_3(\alpha)]; \forall \alpha \vert 0 \leq \alpha \leq 1$ $\begin{cases} c_1(\alpha) = \mathbf{min}(\mathbf{a_1}(\alpha), \mathbf{b_1}(\alpha), \ldots, \mathbf{z_1}(\alpha)) \\ c_3(\alpha) = \mathbf{min}(\mathbf{a_3}(\alpha), \mathbf{b_3}(\alpha), \ldots, \mathbf{z_3}(\alpha)) \end{cases}$	Fuzzy number $\overline{C}(x)$ strictly less than or equal to a given set of fuzzy numbers
CI_j	Critical index of j
CPM	Critical Path Method
EF	Early finish
ES	Early start
$FF_{ij}(p_j \vert w_j \vert z)$	Finish-to-finish relationship
$Fl_{ij}(p_i \vert p_j \vert z)$	Flow relationship
$FS_{ij}(z)$	Finish-to-start relationship
fuzzy-GPSP	Fuzzy project scheduling problem with GPRs
GPRs	Generalized precedence relations
GPSP	Project scheduling problem with GPRs
LF	Latest finish
LS	Latest start
PSP	Project scheduling problem
P6P	Primavera© P6 Professional V8.2.0
RV	Risk value
$SF_{ij}(p_i,w_i \vert p_i,w_i \vert z)$	Start-to-finish relationship
$SS_{ij}(p_i \vert w_i \vert z)$	Start-to-start relationship
TF	Total float