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ORIGINAL RESEARCH

Multiple Sliding Modes Enlarge Basins of Attraction in Switched Control Systems

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Abstract
This paper investigates the role of hidden dynamics in influencing the stability of sliding solutions within control-switched systems. By employing cell-mapping methods, we provide numerical evidence that incorporating hidden dynamics on the switching manifold can extend the sliding dynamics, resulting in a significant expansion of the system's region of attraction. As representative examples, we considered control systems with stable and unstable dynamics converging around multiple equilibrium points.

Keywords Multiple sliding modes · Discontinuous systems · Cell-to-cell- mapping · Hidden dynamics

Introduction

Switching dynamics arise in diverse applications, ranging from electronic circuits [1], biological processes [2], social or economic modeling [3] and mechanical systems [4]. Additionally, dynamical systems can commutate between different operational modes through exogenous control laws such as the well-known Sliding Mode Control [5]. Multiple control techniques have succeeded in ensuring global stability for switched systems; However, there are cases where, due to the system's complexity or large 1 disturbances, the coexistence of unstable, chaotic, periodic, or sliding solutions, among others, can occur [6, 7]. The set of initial conditions that lead to a particular solution is called the basin of attraction which is a measure of how stable the attractor is. Typically, Lyapunov methods based on common and multiple functions are used to estimate these basins of attraction [8, 9]. In [10] the approach of different inclusions is utilized to extend the initial set of the domain of attraction for switched linear systems under saturation. An iterative algorithm based on the sum of squares formulations and Lyapunov functions is used to compute the basins of attraction for discrete-time switched systems [11]. By increasing the distance between equilibria the authors in [12] enlarge the basin of attraction of a multi-stable switching system. In [13] control gains are tunned through the use of the norm of saltation matrix, resulting in a wide stability range for power converters. The hidden dynamics introduced in [4, 14] refer to the behavior over the switching surface governed by nonlinear sliding modes that vanish outside of the switching surface. This approach has been applied to model discontinuities present in rigid body systems, hydraulic systems with valves, climate models, neuronal models and more [15]. Some examples have demonstrated that hidden dynamics can cause solutions to adhere to the switching surface and slide along it, resulting in the emergence of new behaviors [4, 16, 17]. In this paper, we explore the effect of adding hidden terms to the switching control law through the computation of the basin of attraction for a nonlinear system. The document is organized as follows: a brief description of switched control systems is presented in "Switched Control Systems". The incorporation

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of hidden terms into the switching control law is described in "Enlarging Stability of Switched Systems", followed by the computation of the basin of attraction for nonlinear switched systems under hidden dynamics in "Numerical Results", and concluding remarks are provided in "Conclusion".

Switched Control Systems

Switched feedback systems are dynamical systems described as

$$\begin{cases} \dot{x} = f(x) + u \\ u = -\mu \cdot \operatorname{sgn}(h) \end{cases} \tag{1}$$

where $f(x)$ is a nonlinear piecewise continuous vector field having a codimension-one submanifold Σ defined as the zero set of a smooth scalar function $h(x)$. The control signal u is discontinuous with a constant gain μ. Such a system can be rewritten in the Filippov formalism [18] as

$$\dot{x} = f(x) = \begin{cases} f_1(x), & x \in R_1, \\ f_2(x), & x \in R_2, \end{cases} \quad x(0) = x_0 \in \mathbb{R}^n \tag{2}$$

where f_1 and f_2 are smooth vector fields, $R_1 = \{x \in \mathbb{R}^n : h(x) > 0\}$, $R_2 = \{x \in \mathbb{R}^n : h(x) < 0\}$ and $\Sigma = \{x \in \mathbb{R}^n : h(x) = 0\}$. The dynamics on Σ are the well-known sliding if both vector fields f_1 and f_2 point towards Σ, i.e. $\mathcal{L}_{f_1}h(x) \cdot \mathcal{L}_{f_2}h(x) < 0$, $\mathcal{L}_{f_1}h(x) < 0$. The open subset $\hat{\Sigma} \subset \Sigma$ is referred to as the *sliding surface*. Here, $\mathcal{L}_{f_{1,2}}h(x) := \nabla h(x) \cdot f_{1,2}(x)$ is defined as the Lie derivative of $h(x)$ with respect to the vector field $f_{1,2}$. Once the evolution of the system reaches the sliding surface, the dynamics can be defined as a linear combination of the vector fields (f_1 and f_2), namely:

$$\dot{x} = f_\Sigma(x), \quad x \in \hat{\Sigma} \tag{3}$$

where:

$$f_\Sigma(x) = \frac{f_1(x) + f_2(x)}{2} + \frac{f_1(x) - f_2(x)}{2} s(x), \tag{4}$$

and $-1 \le s(x) \le 1$. Since the motion is constrained to $\hat{\Sigma}$, f_Σ must be tangent to $\hat{\Sigma}$, i.e. $\mathcal{L}_{f_\Sigma}h(x) = 0$, which yields

$$s(x) = -\frac{\mathcal{L}_{f_1+f_2}h(x)}{\mathcal{L}_{f_1-f_2}h(x)}. \tag{5}$$

On the other hand, the trajectory of the system leaves the sliding region as soon as the vector fields $f_{1,2}$ become tangent to Σ, that is, when any of these conditions are satisfied: $\mathcal{L}f_1h(x) = 0$ or, $\mathcal{L}f_2h(x) = 0$.

Enlarging Stability of Switched Systems

The term *hidden dynamics* introduced in [14], refers to extending the sliding surface over the crossing regions by adding terms with a nonlinear dependence of s that disappear outside the switching manifold Σ hence they are *hidden* in (2). Considering the switched systems (2), the sliding equations under the hidden dynamics approach reads

$$\begin{aligned} \dot{x} &= f_\Sigma^{nl} \\ f_\Sigma^{nl} &= \frac{1}{2}[1+s]f_1(x) + \frac{1}{2}[1-s]f_2(x) + G(s)\gamma(s) \end{aligned} \tag{6}$$

where $G(s)$ is any nonlinear function of s and $\gamma(s)$ satisfies

$$\gamma(s) \in \begin{cases} 0 & \text{if} \quad |s| = 1 \\ [0,1] & \text{if} \quad |s| < 1 \end{cases}. \tag{7}$$

To determine if such dynamics are attractive or repellent, we can check whether the trajectories outside of $h(x)$ converge toward the sliding surface or not. This can be evaluated through the following relation

$$z(x,s) := \frac{\partial}{\partial s}(f(x,s) \cdot \nabla h(x)) < 0 \tag{8}$$

where (6) is attracting if $z(x, s)$ is negative, and repelling if $z(x, s)$ is positive. Places where there exist sliding motions where the vector field f is tangent to Σ for one or more values of s^*, allowing the flow slides along Σ. The nonlinear sliding region is given by $\hat{\Sigma}^{nl} = \{x \in \Sigma : f(x,s) \cdot \nabla h(x) \neq \mathbf{0}\}$, places where there are no sliding dynamics, define the nonlinear crossing regions, $\Sigma^{nc} = \{x \in \Sigma : f(x,s) \cdot \nabla h(x) = \mathbf{0}\}$ such that, $\Sigma = \hat{\Sigma}^{nl} \cup \Sigma^{nc}$.

Following this idea, the stability of the sliding mode for the system (2) can be extended by rewriting the control action u as

$$u_{ns} = -\mu\left[\operatorname{sgn}(h) + \hat{\alpha}\operatorname{sgn}(h)\left(1 - \operatorname{sgn}(h)^2\right)\right] \tag{9}$$

or

$$u_{ns} = -\mu\left[s + \hat{\alpha}s\left(1 - s^2\right)\right], \quad s = \operatorname{sgn}(h) \tag{10}$$

where $\hat{\alpha}$ is a constant parameter that increases the basins of attraction of the sliding mode. Note that according to (6), $\gamma = 1 - s^2$ and $G = \hat{\alpha}s$.

Numerical Results

To demonstrate how the stability basins change with the incorporation of hidden terms in the discontinuity manifold, we examine two switched control systems. In this section, we compute the region of attraction for equilibrium points located both within and outside of the discontinuity manifold. We vary the parameter $\hat{\alpha}$ and select different hidden terms.

Example 1: Feedback Stabilization

The systems under consideration is a single-input single-output relay feedback system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

where the discontinuous control is given by $u = -\text{sgn}(y)$ which guarantee the system converges to the equilibrium point $x* = (0, 0)$. Numerical solutions are displayed in Fig. 1a and b. It is shown that the system exhibits sliding solutions giving rise to a stable pseudo-equilibrium point $(x*)$ on the switching surface Σ. According to (2) the system reads

$$F_1 = \begin{bmatrix} -x_1 + x_2 \\ 3x_2 + 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -x_1 + x_2 \\ 3x_2 - 1 \end{bmatrix} \tag{12}$$

with $h(x) = x_1 + x_2$ and $s = \text{sgn}(h)$. From (4), the sliding vector field is $f_\Sigma = \begin{bmatrix} -x_1 + x_2 & x_1 - x_2 \end{bmatrix}^T$

Once we determine the vector fields F_1, F_2 and F_Σ we make use of the ESCM algorithm for computing the basins of attraction presented in [7]. This algorithm encompasses a dynamic construction of the cell-state-space therefore, a lower number of integrations are required.

Figure 2 displays the basins of attraction of the equilibrium point $x* = (0, 0)$, where the region colored in yellow belongs to its basin of attraction, while initial conditions in the blue region are mapped outside.

Now, to extend the basins of attraction of the equilibrium point $x*$, we use the control law (10), for $\hat{\alpha} = 20$, i.e.

$$u_{ns1} = -\left[\text{sgn}(h) + 20\text{sgn}(h)\left(1 - \text{sgn}(h)^2\right)\right] \tag{13}$$

Figure 3 indicates an extension of the basins of attraction for the equilibrium point $x*$, for different values of $\hat{\alpha}$. Given that the presence of disturbances can lead the system away

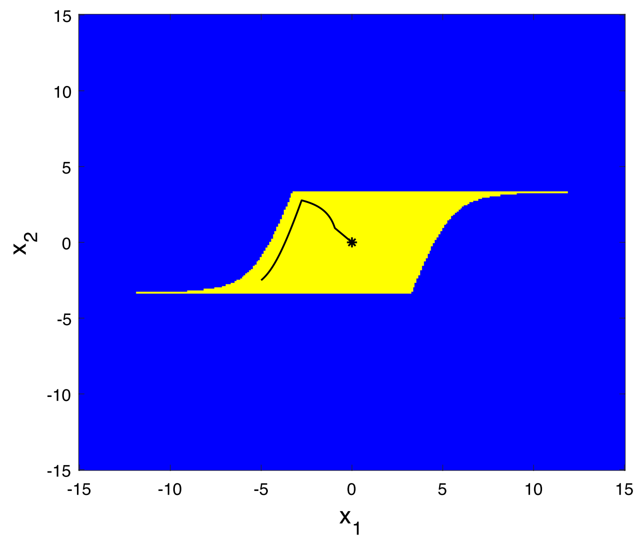


Fig. 2 Basins of attraction for the sliding control problem (11), for an initial resolution of 15×15 grid of cells. Black curves stand for trajectories inside and outside of the BA of $x*$

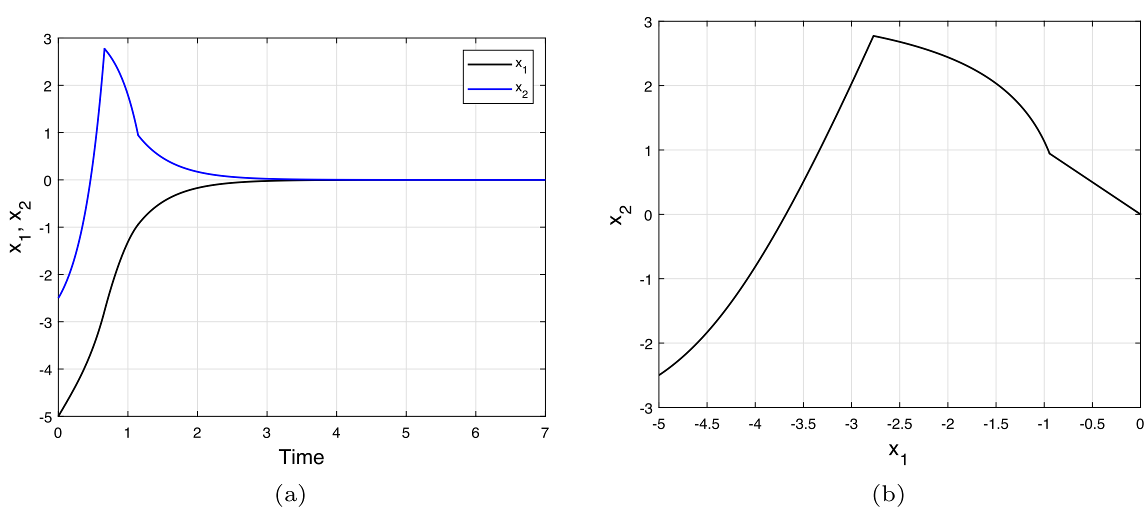


Fig. 1 Numerical simulations of (11) regarding linear sliding modes: **a** time evolution and **b** phase portrait. The trajectory solution starts with initial condition $(x_1, x_2) = (-2.5 \ -5)$

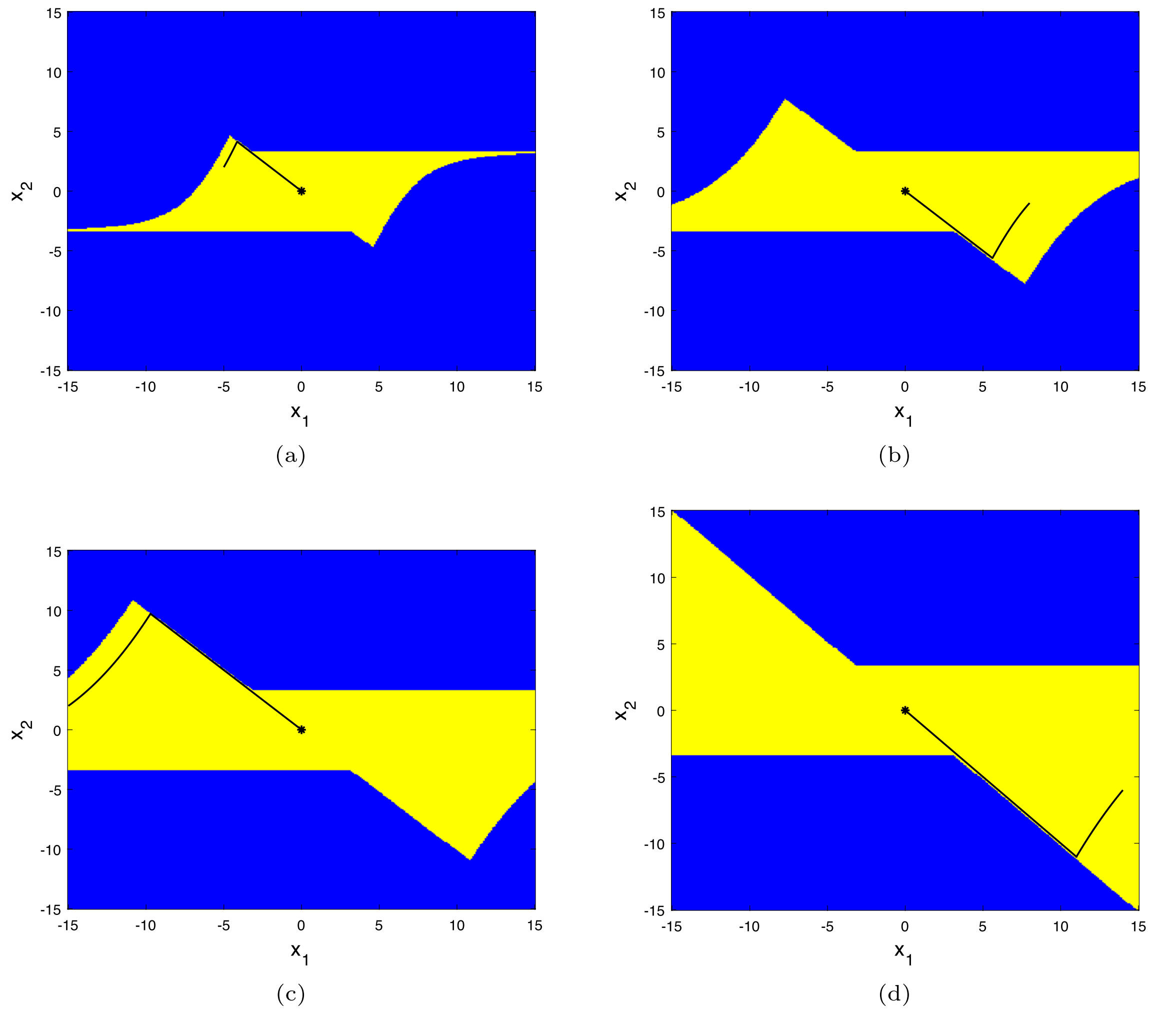


Fig. 3 Basins of attraction for the sliding control problem (11), for an initial resolution of 15×15 grid of cells: **a** $\hat{\alpha} = 4$, **b** $\hat{\alpha} = 8$, **c** $\hat{\alpha} = 12$ y **d** $\hat{\alpha} = 20$

from its equilibrium point [19], numerical results show that by expanding the set of initial conditions for which system trajectories converge to the equilibrium point, the system becomes capable of rejecting extra disturbances.

Note that, we can also extend the basin of attraction of the equilibrium point $x*$, by considering the following function

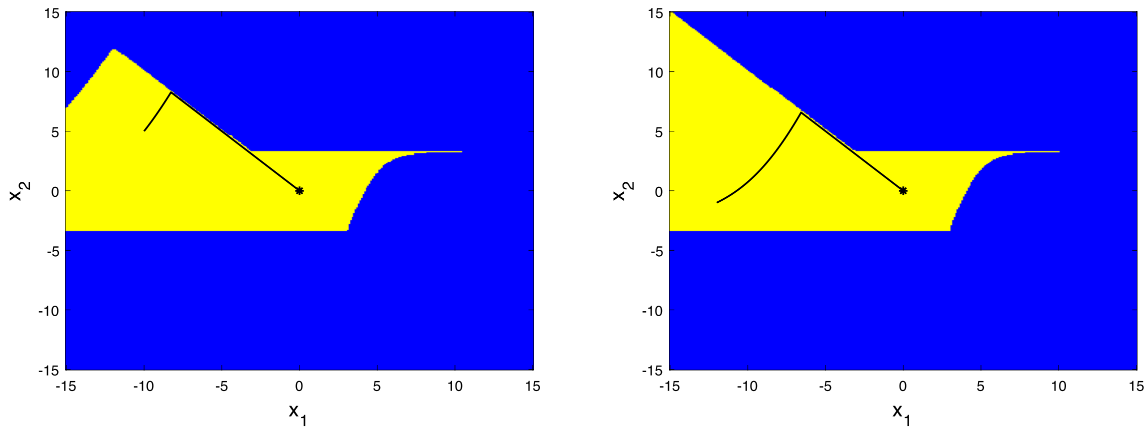
$$u_{ns1} = -\left[\mathrm{sgn}(h) + \hat{\alpha}\mathrm{sgn}(h)^2\left(1 - \mathrm{sgn}(h)^2\right)\right] \tag{14}$$

Figure 4a and b shows how the invariant region is increased, however with this particular selection of the hidden terms, we note that the obtained invariant region is smaller than the previous.

Example 2: System with Non Sliding Dynamics

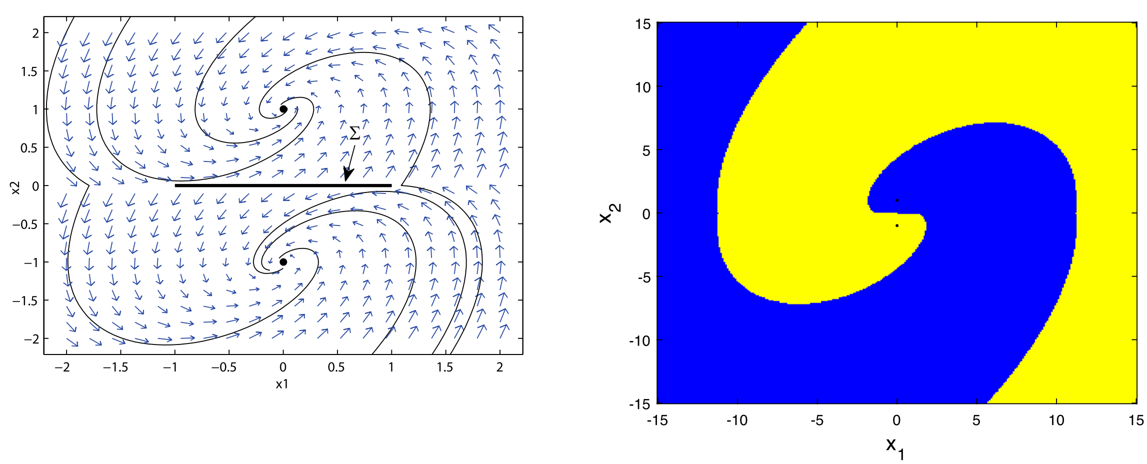
Let us consider a system with unstable sliding solutions collapsing to two stable equilibrium points located outside of the discontinuity manifold, written as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{15}$$



(a) (b)

Fig. 4 Basins of attraction for the system (11) using ESCM, for an initial resolution of 15×15 grid of cells. **a** $\hat{\alpha} = 20$ and **b** $\hat{\alpha} = 50$



(a) (b)

Fig. 5 **a** Phase portrait of the system for $\varepsilon = 0$, with some random trajectories started in the regions separated by the switching surface. **b** Basins of attraction for the system (15)

with $u = \mathrm{sgn}(y) + \varepsilon \cdot \mathrm{sgn}(y)(1 - \mathrm{sgn}(y)^2)$ and ε a constant value. As shown in Fig. 5a, the system presents coexistence of two stable equilibrium points at $x*_1 = (0, 1)$ and $x*_2 = (0, -1)$. Moreover the vector fields $F_1 = \left[-x_2 + 1 \;\; x_1 - x_2 + 1\right]^T$, $F_2 = \left[-x_2 - 1 \;\; x_1 - x_2 - 1\right]^T$ point in opposite direction at the switching manifold, i.e. $\mathcal{L}_{F_1 - F_2}(h)(x) > 0$, therefore we have an unstable sliding surface.

For $\varepsilon > 0$ the repealing area is larger as ε increases, restricting the basin of each equilibrium to one side of the switching manifold Σ, and limiting crossing trajectories as it is displayed in Fig. 6. Note that despite of equilibrium point is not located on the sliding manifold, new regions are incorporated to the basins of attraction of each the stable equilibrium point due to the nonlinear terms on Σ.

Conclusion

The results presented in this paper indicate the importance of hidden dynamics in the stability of the sliding solutions. A particular selection of the switching surface causes the extension of the sliding dynamics, allowing the increase of the system's region of attraction, specifically for larger values of $\hat{\alpha}$. It is worth mentioning that hidden terms can cause multiple sliding dynamics to alternate between

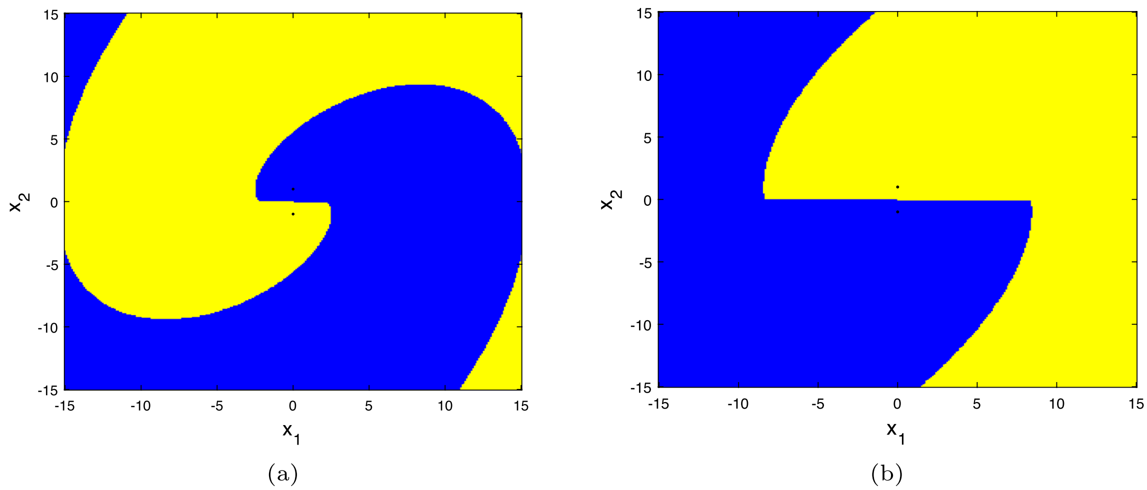


Fig. 6 Basins of attraction for the system (15) with some random trajectories started in the regions separated by the switching surface, **a** $\varepsilon = 4$, **b** $\varepsilon = 20$

attraction and repulsion, leading to an asymmetric shape for the region of attraction. Given that the presence of disturbances can lead the system away from its equilibrium point, expanding the set of initial conditions for which system trajectories converge to the equilibrium point makes the system capable of rejecting large disturbances.

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Code Availability Not applicable.

Declarations

Conflict of interest The author's declare that they have no conflict of interest.

Ethics Approval and Consent to Participate Not applicable.

Consent for Publication Not applicable.

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