Rubem P. Mondaini *Editor*

Trends in Biomathematics: Chaos and Control in Epidemics, Ecosystems, and Cells

Selected Works from the 20th BIOMAT Consortium Lectures, Rio de Janeiro, Brazil, 2020

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Viability Analysis of Labor Force in an Agroforestry System

I. M. Cholo Camargo, J. A. Amador Moncada, C. A. Peña Rincón, and G. Olivart Tost

1 Introduction

Societies in general incorporate their environment that includes supporting ecosystems as an essential part for economic growth that allows them to offer well-being to their population. For this purpose, strategies that frame the protection of natural resources have been developed, for example, through the common pool resource (CPR) theory [1]. However, the results of the implementation of these strategies are not fully achieved in practice[2]. Moreover, the dynamic complexity of the relationships between different economic, governmental, and natural systems proposes great challenges to the agro-ecological systems that are essential to understand food production as well as the challenges of environmental impacts and their consequences on the environment [3–5]. On the other hand, socio-ecological systems coupled nature and humans, which recognize people as part and not part

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of nature [6], where the dynamics of the social system influence an ecological system and vice versa [7], encouraging to lead scenarios for natural resources to be overexploited [8, 9].

The most affected natural resources are found in forests on a global scale, and they are deforested mainly by the change in land use due to agriculture, timber production, and infrastructure investment [10]; most deforestation occurs in the tropics [11, 12]. Therefore, this problem has been addressed from the socio-ecological modeling by making models that consider several ecological dynamic configurations thus allowing to analyze different types of social pressure on resources [13]. Additionally, models that include technology to overcome the increase in the scarcity of renewable resources have been considered [14]; models that take into account economic growth fluctuations motivated by dynamic resources have also been studied [15, 16]. A very inclusive model was proposed by Brander and Taylor [15], often called BT-type models, characterized mainly by the following attributes: (1) population growth, (2) substitutability, (3) innovation, (4) capital accumulation, (5) property rights and conservation policies for renewable and non-renewable resources, and (6) modeling approach with the purpose of understanding the dynamics between society and natural resources by including renewable resources, agriculture, and a manufactured good [17].

In the 2D model proposed by DAlessandro [16], two important considerations are included: agriculture as an economic activity parallel to resource extraction and a critical level of resources in which its regeneration rate becomes negative and, therefore, total resource depletion becomes inevitable [18]. Results in that paper are basically phase portraits that show the change in the vector field of the system under parameter changes. Other works analyzed the model from the bifurcation theory in order to characterize all the possible steady-state scenarios in the parametric space [19–22], showing that adequate parameter sets can lead to sustainable trajectories where population and resource level are higher than zero in the long run.

In this chapter, we modify the model in [16] to perform the analytical study of its steady-state dynamics and approached it from the viability theory to evaluate the trajectory under some restrictions that guarantee a minimum value for the state variable at each instant of time t. This approach focuses on the transient state of the system to identify the set of thresholds associated with the system solution that ensures the coexistence of the social and ecological component rather than pointing to steady-state analysis, see [23, 24].

The chapter is organized as follows. In Sect. 2, we introduce some description of DAlessandro's model [16] and its purpose and, additionally, the modification and its advantages to the local dynamic analysis. In Sect. 3, we define some concepts regarding viability theory and their application to the modified model. In Sect. 4, an analysis was made of the numerical simulations performed, which reflect the theoretical results. We conclude in Sect. 5, by discussing the exposed results and by giving our contributions and some topics considered as challenges.

2 Mathematical Model

The mathematical model proposed in this chapter for the viability analysis is a modification to DAlessandro's system [16], which considers a population that obtains its livelihoods mainly from the exploitation of renewable resources, as well as from its agricultural activities within its territory. The following is an introduction to the generalities of the base model [16]. In addition, the modification made is shown to facilitate the application of analytical methods.

2.1 Base Model

The base model presents a system of ordinary differential equations (ODEs) representing the dynamic interaction between the population growth and the exploitation of available renewable resources in an isolated society. It is assumed that the dynamics of the population depends on the income acquired by the economic activities within its territory, being agriculture represented by a Cobb–Douglas production function and the exploitation of the natural resource that must satisfy the minimum basic needs required by the population. The model assumes that the forest grows logistically and decreases according to a Schaefer production function [25]. This model has been studied from the perspective of bifurcation analysis in order to understand the behavior of steady-state variables [21, 22, 26]. The coupled system of ODEs for the base model is

$$\begin{cases} \dot{L} = \gamma \left(\alpha_1 \left(1-\beta\right)^{\delta} L^{\delta-1} + \phi\alpha_2\beta S - \sigma\right) L, \\ \dot{S} = \left[\rho\left(\frac{S}{k_2}-1\right)\left(1-\frac{S}{k_1}\right) - \alpha_2\beta L\right] S, \end{cases} \tag{1}$$

where

- L is the level of population.
- S is the level of renewable resources available.
- β is the proportion of people dedicated to the exploitation of the resource. Thus, $1-\beta$ is the proportion of population dedicated to agriculture.
- ρ represents the natural growth rate of the forest.
- k_1 is the carrying capacity of the resource.
- k_2 is a threshold where the forest growth rate becomes negative, and therefore, the depletion of the resource is inevitable; this is called the strong Allee effect.
- ϕ and γ are the representative values of the resource and agricultural products in terms of calories, respectively.
- α_1 is a measure of the productivity of the land.

- α_2 is a measure of effectiveness in resource extraction.
- σ is the minimum value of per capita income in terms of calories necessary to survive.
- δ is the elasticity of the Cobb–Douglas function.

2.2 Modified Model

The base model (1) presents equilibrium points with a value of $L = 0$, where it was not feasible to determine its local behavior analytically by means of the linear approximation. Furthermore, an explicit expression of the internal equilibrium points was not determined in the positive quadrant of the system. To overcome these difficulties, we propose a quadratic approximation of Lagrange in the term L^δ assuming a maximum population level of 5000.

$$L^\delta \approx g(L) = aL^2 + bL, \tag{2}$$

where

$$a = -5000^{\delta-2}(1 - 2^{\delta-1}) < 0 \text{ y } b = 5000^{\delta-1}(2 - 2^{\delta-1}) > 0.$$

The modified model incorporating Eq. (2) is

$$\begin{cases} \dot{L} &= \gamma\big[\alpha_1(1-\beta)^\delta g(L) + \phi\alpha_2\beta LS - \sigma L\big]. \\ \dot{S} &= \rho S\left(\frac{S}{k_2} - 1\right)\left(1 - \frac{S}{k_1}\right) - \alpha_2\beta LS, \\ & L(t) \geq 0 \;\; S(t) \geq 0. \end{cases} \tag{3}$$

2.3 Equilibrium Points

The modified system presents five or six equilibrium points, of which up to two are in the positive quadrant and are characterized by $L > 0$ and $S > 0$, these being the points of interest for this chapter. By using the definition of equilibria of a system of ODEs and performing algebraic operations, a quadratic equation for S of the form $AS^2 + BS + C = 0$ is obtained, where

$$A = a\alpha_1(1-\beta)^\delta\rho,$$

$$B = -A(k_1 + k_2) - k_1k_2\phi\alpha_2^2\beta^2,$$

$$C = k_1k_2\alpha_2\beta\left(\sigma - b\alpha_1(1-\beta)^\delta\right) + a\alpha_1(1-\beta)^\delta\rho k_1k_2.$$

If it is defined that

$$\Delta = \alpha_2^2\beta^2\phi k_1k_2 + a\alpha_1k_1\rho(1-\beta)^\delta + a\alpha_1k_2\rho(1-\beta)^\delta,$$

the possible equilibrium points are

$$(L_1.S_1),\ \ (L_2.S_2),\ \text{y}\ (L_{00}, S_{00}),$$

where

$$S_1 = \frac{\Delta + \sqrt{B^2 - 4AC}}{2a(1-\beta)^\delta\alpha_1\rho}, \tag{4}$$

if is satisfied that

$$\alpha_2^2\beta^2\phi k_1k_2 + \sqrt{B^2 - 4AC} < -a\alpha_1\rho(1-\beta)^\delta\left(k_1 + k_2\right),$$

and

$$S_2 = \frac{\Delta - \sqrt{B^2 - 4AC}}{2a(1-\beta)^\delta\alpha_1\rho}, \tag{5}$$

if it is satisfied that

$$\alpha_2^2\beta^2\phi k_1k_2 < -a\alpha_1\rho(1-\beta)^\delta\left(k_1 + k_2\right) + \sqrt{B^2 - 4AC}.$$

By replacing the values of S obtained in (4) and (5) in (6), coordinates of $L > 0$ for the internal equilibrium points are obtained

$$L_{1,2} = \frac{\sigma - b\alpha_1(1-\beta)^\delta - \phi\alpha_2\beta S_{1,2}}{a\alpha_1(1-\beta)^\delta}, \tag{6}$$

as long as it is true that

$$\sigma < b\alpha_1(1-\beta)^\delta + \phi\alpha_2\beta S_{1,2}.$$

These two internal points exist when the discriminant $B^2 - 4AC > 0$. If the discriminant is zero, we obtain that

$$S_{00} = \frac{\Delta}{2a(1-\beta)^\delta\alpha_1\rho}$$

with the restriction

$$\alpha_2^2\beta^2\phi k_1k_2 < -a\alpha_1\rho(1-\beta)^\delta (k_1+k_2).$$

Finally, the value of L for S_{00} is obtained

$$L_{00} = \frac{\sigma - b\alpha_1(1-\beta)^\delta - \phi\alpha_2\beta S_{00}}{a\alpha_1(1-\beta)^\delta}.$$

3 Viability

A qualitative analysis for the base model (3) shows the existence of stationary state scenarios in which the population can consume natural resources without completely depleting them, thus guaranteeing their own existence. This behavior is possible when the trajectory converges either to an internal equilibria or to a limit cycle, depending on the configuration of the system parameters and the initial conditions, see, for example, [22].

Now, if the purpose is to evaluate the trajectory under some restrictions that guarantee a minimum value for the state variables at each instant of time t, it is necessary to use the viability theory. From [27], viability is defined as the capacity for a system to maintain conditions of existence through time given certain constraints; the set of initial conditions for which a control exists in such a way that the system satisfies and complies with the restrictions is called the viability kernel. In practice, the initial conditions cannot easily be controlled, while the restrictions are more likely to be modified. Therefore, an equivalent definition of viability kernel is the set of thresholds values for which the dynamical system (7) becomes compliant with the constraints (8) during a period of time by applying an admissible control strategy given a certain initial condition [28]. This set of thresholds is called sustainable thresholds. We use this last definition for viability analysis in the work.

3.1 Preliminary

Consider the following system:

$$\begin{cases} \dot{x}(t) = f(x(t), \mu(t)) \\ x(0) = x_0 \\ \mu(t) \in [\mu_{min}, \mu_{max}], \end{cases} \tag{7}$$

where $x(t) \in \Re^n$ is the vector of state variables and $\mu(t) \in \Re$ is the control variable that represents a decision that would be applied to the system.

For the system (7) to have a solution and to be unique, it is assumed that $f(x(t), \mu(t))$ is locally Lipschitz.

Depending on the initial condition, additional conditions (8) may be required to be met, which shall be considered as system constraints of (7).

$$I_j(x(t), \mu(t)) \geq \theta_j \quad j = 1, \ldots p. \tag{8}$$

Therefore, we have the following definition.

Definition 3.1 The set of sustainable thresholds of the system (7) with control $\mu(t)$ is given by

$$U_{t^*}(x_0) = \left\{\theta | \exists \left(x\left(t\right), \mu\left(t\right)\right), \text{satisfies } (7) \; and \; (8) \; \forall t \in \left(0, t^*\right)\right\}. \tag{9}$$

Schaefer [25] expresses the dynamics of a renewable resource in terms of its natural growth $G(S)$ and its exploitation $h(S)$ as: $\dot{S} = G(S) - h(S)$. By varying a factor in such a dynamics that affects the harvesting of the resource, a different trajectory is obtained; for example, if the extraction workforce increases, renewable resources will disappear faster. Theorem 3.1 is motivated from the above, where each Cauchy problem refers to the resource dynamics with a different control.

Theorem 3.1 *Let $f : A \to \Re$ be a continuous function with $A \subseteq \Re \times \Re$ open. Let us suppose that there is a function $g : \Re \times \Re \to \Re$ that is continuous and locally Lipschitz in $x(t)$ such that $f(x, \mu) \leq g(x, \mu)$ for all $(x, \mu) \in A$, $\frac{\partial g}{\partial \mu} \leq 0$, where $\mu(t) \in \Re$ is the control variable. Moreover, let $(x_0, \mu_1(t_0))$, $(y_0, \mu_2(t_0)) \in A$ be two initial conditions, such that $x_0 \leq y_0$ and $\mu_1(t) \geq \mu_2(t)\, \forall t \geq t_0$. if $x(t)$ and $y(t)$ are the solutions of the following Cauchy problems:*

$$\begin{aligned} \dot{x}(t) &= f(x(t), \mu_1(t), t) & \qquad \dot{y}(t) &= g(y(t), \mu_2(t), t) \\ x(t_0) &= x_0 & \qquad y(t_0) &= y_0; \end{aligned}$$

then

$$x(t) \leq y(t) \;\; \forall t \in I.$$

I is the solution existence interval for $x(t)$ and $y(t)$.

Proof *(**Contradiction**)* Let us suppose that there exists $\bar{t} \geq t_0, \bar{t} \in I$, such that $x(\bar{t}) > y(\bar{t})$.

Then, we define t^* by

$$t^* = \sup\{t \in [t_0, \bar{t}] \;\mid\; x(t) \leq y(t)\},$$

by the continuity of the solutions $x(.)$ e $y(.)$, we have $x(t^*) = y(t^*)$. On the other hand, f is locally Lipschitz in $x(t^*)$, hence, the constants $\epsilon_1 > 0$ and $L_f \geq 0$ will exist, such that $|f(x, s) - f(y, s)| \leq L_f|x - y|$ for every $x, y \in (x(t^*) - \epsilon_1, x(t^*) + \epsilon_1)$ and for every $s \geq 0$. Moreover, as $t^* < t - a$ new constant $\epsilon_2 > 0$ appears, such that $t^* + \epsilon_2 < t-$. Finally, for every $s \in (t^*, t^* + \epsilon_2)$, $x(s), y(s) \in (x(t^*) -$

$\epsilon_1, x(t^*) + \epsilon_1)$. Thus, for every $s \in (t^*, t^* + \epsilon_2)$, it is satisfied that $|f(x(s), s) - f(y(s), s)| \leq L_f |x(s) - y(s)|$. Let us define a continuous function $w(t)$ as $w(t) = x(t) - y(t) > 0\ \forall t \in (t^*, t^* + \epsilon_2]$; then,

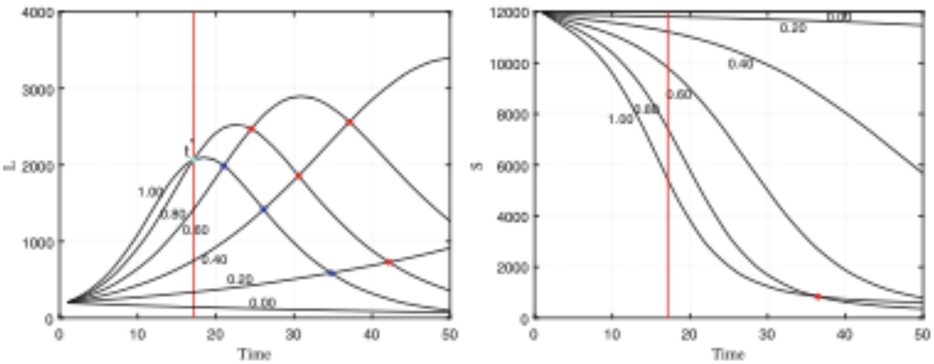
$$
\begin{aligned}
0 < w(t) &= x(t) - y(t) \\
&= x_0 + \int_{t_0}^{t} f(x(s), \mu_1(s), s) ds - y_0 - \int_{t_0}^{t} g(y(s), \mu_2(s), s) ds \\
&= x_0 + \int_{t_0}^{t^*} f(x(s), \mu_1(s), s) ds + \int_{t^*}^{t} f(x(s), \mu_1(s), s) ds - y_0 \\
&\quad - \int_{t_0}^{t^*} g(y(s), \mu_2(s), s) ds - \int_{t^*}^{t} g(y(s), \mu_2(s), s) ds \\
&= x(t^*) + \int_{t^*}^{t} f(x(s), \mu_1(s), s) ds - y(t^*) - \int_{t^*}^{t} g(y(s), \mu_2(s), s) ds \\
&= \int_{t^*}^{t} f(x(s), \mu_1(s), s) ds - \int_{t^*}^{t} g(y(s), \mu_2(s), s) ds \\
&\leq \int_{t^*}^{t} g(x(s), \mu_1(s), s) ds - \int_{t^*}^{t} g(y(s), \mu_2(s), s) ds \qquad (f(x, \mu) \leq g(x, \mu)) \\
&\leq \int_{t^*}^{t} g(x(s), \mu_2(s), s) ds - \int_{t^*}^{t} g(y(s), \mu_2(s), s) ds \qquad \left(\frac{\partial g}{\partial \mu} \leq 0, \quad \mu_1 \geq \mu_2 \right) \\
&= \int_{t^*}^{t} g(x(s), \mu_2(t), s) - g(y(s), \mu_2(s), s) ds \\
&\leq L_g \int_{t^*}^{t} |x(s) - y(s)| ds \\
&= L_g \int_{t^*}^{t} |w(s)| ds \\
&= L_g \int_{t^*}^{t} w(s) ds.
\end{aligned}
$$

From Gronwall's lemma, it is concluded that $w(t) = 0$, which contradicts the approach that $w(t) > 0$, thus $x(t) \leq y(t)\ \ \forall t \in I$. □

If two different controls are applied starting from the same initial point, the corresponding trajectories will be kept separated during a time interval I, because it may happen that there is a different crossing from the initial condition at a time t, as seen in Fig. 1. Proposition 3.1 demonstrates the existence of this interval I.

Proposition 3.1 *Let $f, g : A \to \Re$ be continuous functions with $A \subseteq \Re^2$ open. Consider the following system of ordinary differential equations:*

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), t) \\
\dot{y}(t) &= g(y(t), t),
\end{aligned}
$$



(a) (b)

Fig. 1 Monotonicity of the state variables in the interval $(0, t^*)$ and $(L_0, S_0) = (200, 12000)$

such that $x(0) = y(0) = \bar{x}$ *y* $f(\bar{x}, 0) < g(\bar{x}, 0)$. *Then, there is an interval* I *where*

$$x(t) < y(t) \quad \forall t \in I. \tag{10}$$

Proof Let be $z = x - y$; then,

$$\begin{aligned} z(0) &= x(0) - y(0) \\ &= \bar{x} - \bar{x} \\ &= 0. \end{aligned}$$

On the other hand, $\dot{z}(t) = f(x(t), t) - g(y(t), t)$, and thus

$$\begin{aligned} \dot{z}(0) &= f(x(0), 0) - g(y(0), 0)) \\ &= f(\bar{x}, 0) - g(\bar{x}, 0) < 0, \end{aligned}$$

but

$$\begin{aligned} \dot{z}(0) &= \lim_{t \to 0} \frac{z(t) - z(0)}{t} \\ &= \lim_{t \to 0} \frac{z(t)}{t} < 0, \end{aligned}$$

which implies that $z(t) < 0$, that is, $x(t) - y(t) < 0$, thus, by the sign conservation theorem, there is a $\epsilon > 0$ such that $x(t) < y(t)$ for every $t \in (0, \epsilon)$. □

If we look at the solution of the system (3) in Fig. 1, it can be seen that there is an interval I where there is a monotonicity of the trajectories for both L and S, taking different controls of $0 \leq \beta \leq 1$. The interval corresponding to the state variable L

is contained in the interval associated with the state variable S; in other words, the monotonicity of L implies the monotonicity of S, which leads to Theorem 3.2.

Theorem 3.2 *Let $f : A \to \Re$ be a continuous and locally Lipschitz function with $A \subseteq \Re$ open. Suppose that there are two controls $\mu_1(t)$ and $\mu_2(t)$ such that $\mu_1(t) \geq \mu_2(t)$ and $L_1(t) \geq L_2(t)$ for all $t \geq t_0$, where $L_i(t)$ is the fixed solution of $\dot{L}(t)$ with the respective control. Consider the solutions $x(t)$ and $y(t)$ of the following problems:*

$$\dot{x}(t) = f(x(t)) - \mu_1(t)L_1(t)x(t) \qquad \dot{y}(t) = f(y(t)) - \mu_2(t)L_2(t)y(t)$$
$$x(t_0) = x_0 \qquad y(t_0) = y_0$$

with $x_0 = y_0$.

Then, $x(t) \leq y(t)$ for all $t \in I$, where I is the solution existence interval.

Proof **(Contradiction)** Suppose there is a $\bar{t} > t_0$, $\bar{t} \in I$ such that $x(\bar{t}) > y(\bar{t})$.

Then, t is defined as

$$t^* = \sup\{t \in [t_0, \bar{t}) \mid x(t) \leq y(t)\}.$$

Now, $x(\cdot)$ and $y(\cdot)$ are continuous in t^*; then, $x(t^*) = y(t^*)$, f is locally Lipschitz in $x(t^*)$, and hence there is $\epsilon_1 > 0$, $L_f \geq 0$ such that $|f(x) - f(y)| \leq L_f|x - y|$ for every $x, y \in (x(t^*) - \epsilon_1, x(t^*) + \epsilon_1)$ and for every $s \geq 0$. In addition, by definition of t^*, $t^* < \bar{t}$, and thus there is $\epsilon_2 > 0$ such that $t^* + \epsilon_2 < \bar{t}$, and for every $s \in (t^*, t^* + \epsilon_2)$, we have $x(s), y(s) \in (x(t^*) - \epsilon_1, x(t^*) + \epsilon_1)$. Thus, for every $s \in (t^*, t^* + \epsilon_2)$, we have $|f(x(s)) - f(y(s))| \leq L_f|x(s) - y(s)|$. Let us define a continuous function $w(t)$ as $w(t) = x(t) - y(t) > 0\ \forall t \in (t^*, t^* + \epsilon_2]$; then,

$$\begin{aligned}
0 < w(t) &= x(t) - y(t) \\
&= x_0 + \int_{t_0}^{t} f(x(s)) - \mu_1(s)L_1(s)x(s)ds - y_0 - \int_{t_0}^{t} f(y(s)) - \mu_2(s)L_2(s)y(s)ds \\
&= x_0 + \int_{t_0}^{t^*} f(x(s)) - \mu_1(s)L_1(s)x(s)ds + \int_{t^*}^{t} f(x(s)) - \mu_1(s)L_1(s)x(s)ds - y_0 \\
&\quad - \int_{t_0}^{t^*} f(y(s)) - \mu_2(s)L_2(s)y(s)ds - \int_{t^*}^{t} f(y(s)) - \mu_2(s)L_2(s)y(s)ds \\
&= x(t^*) + \int_{t^*}^{t} f(x(s)) - \mu_1(s)L_1(s)x(s)ds - y(t^*) - \int_{t^*}^{t} f(y(s)) - \mu_2(s)L_2(s)y(s)ds \\
&= \int_{t^*}^{t} f(x(s)) - \mu_1(s)L_1(s)x(s)ds - \int_{t^*}^{t} f(y(s)) - \mu_2(s)L_2(s)y(s)ds \\
&= \int_{t^*}^{t} f(x(s)) - \mu_1(s)L_1(s)x(s) - f(y(s)) + \mu_2(s)L_2(s)y(s)ds \\
&\leq \int_{t^*}^{t} f(x(s)) - \mu_1(s)L_1(s)x(s) - f(y(s)) + \mu_1(s)L_1(s)x(s)ds
\end{aligned}$$

$$= \int_{t^*}^{t} f(x(s)) - f(y(s))ds$$
$$\leq L_f \int_{t^*}^{t} |x(s) - y(s)|ds$$
$$= L_f \int_{t^*}^{t} |w(s)|ds$$
$$= L_f \int_{t^*}^{t} w(s)ds.$$

From Gronwall's lemma, it is concluded that $w(t) = 0$, which contradicts the approach that $w(t) > 0$. Thus, $x(t) \leq y(t) \ \forall t \in I$. □

Solow [29] studied the sustainability in an economic model of consumption–production of non-renewable resources, and he found an expression for maximal sustainable consumption without considering a restriction on the conservation of the resource, while Martinet in [30] considered the restriction and found a relationship between a guaranteed minimum consumption and the minimum reserve of the resource. There is something similar for the case under study. An important factor in the exploitation of a renewable resource is the labor force that allows to obtain enough from the natural environment, thus guaranteeing the livelihood of the population and a sustainable level of resources. With the above, there is the minimum level of resources that ensure livelihood with a minimum level of population given. This relationship can be generalized by using Proposition 3.2 where $p - 1$ constraints are set to find the sustainable threshold θ_p.

Proposition 3.2 *Consider a system of the form* (7) *and the constraints* (8)*. For a given initial condition, the set of sustainable thresholds $\mathscr{U}_{t^*}(x_0)$ that depends on $p - 1$ guaranteed constraints is given by*

$$\mathscr{U}_{t^*}(x_0) = \{(\theta_1, \ldots, \theta_p) \mid \theta_1, \theta_2, \ldots \theta_{p-1} \geq 0 \ \theta_p \leq \theta_p^+(\theta_1, \ldots \theta_{p-1})\},$$

where $\theta_p^+(\theta_1 \ldots \theta_{p-1}) := \max\{\theta \mid \exists \ \mu(t), x(t)$ *starting from* x_0*, such that satisfy* (7) *and* $I_j(x(t), \mu(t)) \geq \theta_j, \quad j = 1, \ldots p \ \forall t \in (0, t^*)\}$

Proof $(\Rightarrow)$

Let be $A = \{(\theta_1, \ldots, \theta_p) | \theta_1, \theta_2, \ldots \theta_p - 1 \geq 0$ and $\theta_p \leq \theta_+ p(\theta_1, \ldots \theta_p - 1)\}$. Let us suppose that $\theta \leq \mathscr{U} t^*(x_0)$. Therefore, there exists one control $\mu(t)$ and one path $x(t)$ starting from x_0 satisfying both the dynamics (7) and the restriction $I_j(x(t), \mu(t)) \geq \theta_j$ for all $j = 1, \cdots, p$ at any $t \in (0, t^*)$. Now as, $\theta_j \geq 0 \forall_j = 1 \cdots p - 1$ and $\theta_p \leq \theta + p$, since, is the maximun of the θ_j that satisfy (7) and (8) then $\theta \in A$. By definition, $\theta_j \geq 0 \ \ \forall j = 1 \cdots p - 1$, and from the same definition of $\theta_p^+, \theta_p \leq \theta_p^+$ since θ_p complies with the characteristics to be in the set of which θ_p^+ is the maximum.

$(\Leftarrow)$

Let us consider $\theta_j \geq 0$ with $j = 1 \cdots p - 1$ and $\theta_p \leq \theta_+ p(\theta_1, \ldots \theta_p - 1)$, and suppose that $\theta + p$ is one of the sustainable thresholds, which means that there is one control $\mu(t)$ and one path $x(t)$ starting from x_0 and satisfying both the dynamics (7) and the constraint $Ip(x, \mu) \geq \theta + p\, \forall t \in (0, t^*)$. Now, as $\theta_p \leq \theta + p$ then $Ip(x, \mu) \geq \theta_p$, hence, $\theta \in \mathscr{U} t^*(x_0)$.

□

3.2 *Sustainable Thresholds*

3.2.1 One-Dimensional Case

Consider the following Cauchy problem:

$$\begin{cases} \dot{S}(t) = f(S(t), \beta) \\ S(0) = S_0 \\ \beta \in [0, 1] \end{cases} \tag{11}$$

with the condition

$$S(t) \geq \underline{S} > 0 \ \ \forall t \in (0, t^*), \tag{12}$$

and $f(S(t), \beta)$ is the expression given by the system (3) for S. By Definition 3.1, we have that given an initial condition S_0, the set of sustainable thresholds for the problem (11) with restriction (12) is

$$\begin{aligned} \mathscr{U}_{t^*}(S_0) &= \{\underline{S} \mid \exists (S(\cdot), \beta(\cdot)) \ \text{ such that } \ S(t) \geq \underline{S}\} \\ &= \{\underline{S} \mid \exists (S(\cdot), \beta(\cdot)) \ \text{ such that } \ \underline{S} \leq \hat{S}\}. \end{aligned}$$

Since S is decreasing with respect to the β control, the set of sustainable thresholds is given by Proposition 3.3.

Proposition 3.3 *Let S_0 be an initial condition; then,*

$$\mathscr{U}_T(S_0) = (-\infty, \hat{S}]$$

with

$$\hat{S} = \inf_{t \geq 0} S^*(t),$$

and $S^*(t)$ is the solution of the system with the maximum control β.

Proof $(\Leftarrow)$

Let $\underline{S} \in (-\infty, \hat{S}]$; then, $\underline{S} \leq \hat{S}$, a control that maintains this relationship needs to be found. As $0 \leq \beta \leq 1$, if $\beta = 0$ implies that

$$\dot{S}^* = \rho S^* \left(\frac{S^*}{k_2} - 1\right)\left(1 - \frac{S^*}{k_1}\right). \tag{13}$$

Thus, $\hat{S} \leq S^*(t)$ and by hypothesis $\underline{S} \leq \hat{S}$, hence $\underline{S} \leq S^*(t)$, thus $\underline{S} \in \mathscr{U}_{t^*}(S_0)$.

($\Rightarrow$)
Let be $\underline{S} \in \mathscr{U}_{t^*}(S_0)$; then, there is a β such that (11) is satisfied and $\underline{S} \leq S(t)$ $\forall t \in (0, t^*)$. It is known that $0 \leq \beta \leq 1$, let be $\beta = 0$; then, we have (13), since f is decreasing with respect to the control $S^*(t) \geq S(t)$ $\forall t \in (0, t^*)$ (Theorem 3.1), and by the assumption $\underline{S} \leq S(t)$, we then conclude $\underline{S} \leq S^*(t)$ consequently $\underline{S} \leq \hat{S}$.

□

3.2.2 Two-Dimensional Case

Consider the system (3) in the form (7)

$$\begin{cases} \dot{L} & = \gamma\left[\alpha_1(1-\beta)^\delta g(L) + \phi\alpha_2\beta LS - \sigma L\right] \\ \dot{S} & = \rho S\left(\frac{S}{k_2} - 1\right)\left(1 - \frac{S}{k_1}\right) - \alpha_2\beta LS \\ & \beta \in [0, 1] \end{cases} \tag{14}$$

with the restrictions

$$L(t) \geq \underline{L} \geq 0 \;\; S(t) \geq \underline{S} \geq 0 \;\; \forall t \in (0, t^*). \tag{15}$$

Using Proposition 3.2, sustainable thresholds $\mathscr{U}_{t^*}$ are defined. For this, two restrictions are established, one for the level of people and the other for the forestry resource reserve, that is, the restrictions (15).

After that, Definition 3.1 of the set of sustainable thresholds $\mathscr{U}_{t^*}(L_0, S_0)$ is applied for the reserve level of the resource $\underline{S}$ and of people $\underline{L}$ given an initial condition (L_0, S_0).

Definition 3.2 For a given initial condition (L_0, S_0) of the system (14) with constraints (15), we have

$$\begin{aligned} \mathscr{U}_{t^*}(L_0, S_0) = \{(\underline{L}, \underline{S}) \mid \exists\ \beta(\cdot) \text{ satisfying dynamics (14) and constraints} \\ L(t) \geq \underline{L},\quad S(t) \geq \underline{S}\ \ \forall t \in (0, t^*)\}. \end{aligned}$$

To determine this set, we considered the maximum $\underline{S}$ that can be sustained given $\underline{L}$.

Definition 3.3 Given an initial condition (L_0, S_0) and a minimum reserve level of people $\underline{L}$, $S^+(L_0, S_0, \underline{L})$ is defined as

$$S^+(L_0, S_0, \underline{L}) := \max\{\underline{S} \geq 0 \mid \exists\ \beta(\cdot) \text{ satisfying dynamics (14) and}$$
$$\text{constraints } L(t) \geq \underline{L},\ \ S(t) \geq \underline{S}\ \forall t \in (0, t^*)\}.$$

Thus, this set is characterized by Proposition 3.4, which is obtained by applying Proposition 3.2 to the system (14).

Proposition 3.4 *Let* (L_0, S_0) *be an initial condition of the system* (14)*.*

$$\mathscr{U}_{t^*}(L_0, S_0) = \{(\underline{L}, \underline{S}) \mid \underline{S} \leq S^+(L_0, S_0, \underline{L})\ \ \underline{L} \geq 0\ \ \forall t \in (0, t^*)\}.$$

Proof See the proof of Proposition 3.2. □

Note that a condition for $\underline{L}$ to be a sustainable threshold $\underline{L} \leq L_0$ must be met, since otherwise, from the beginning the sustainability restriction is violated. In addition, there is no control whose associated trajectory is above $\underline{L}$, which is why it is defined $L^+(L_0, S_0)$.

Definition 3.4 Let (L_0, S_0) be an initial condition; then,

$$L^+(L_0, S_0) = \max\{\underline{L} \geq 0 \mid \exists\ \beta(\cdot) \text{ such that } L(t) \geq \underline{L}\ \forall t \in (0, t^*)\}.$$

Numerical simulations in Fig. 1 show that the trajectory $L(t)$ associated to the maximum control $\beta = 1$ is increasing, namely, $L + (L_0, S_0) = L_0$. Now, if trajectory $L(t)$ is increasing for any control $\beta \in [0, 1]$, $L + (L_0, S_0)$ is characterized by Proposition 3.5.

Proposition 3.5 *If an initial condition* (L_0, S_0) *is considered, then*

$$L^+(L_0, S_0) = \min_{t \in (0,t^*)} L_{\beta=1}(t).$$

Proof Let be $A = \{\underline{L} \geq 0 \mid \exists\ \beta(\cdot) \text{ such that } L(t) \geq \underline{L}\ \forall t \in (0, t^*)\}$.

$(\leq)$

Let us suppose that $\underline{L} \in A$. Then, there exists one control $0 \leq \beta \leq 1$ such that $L_\beta \geq \underline{L}$ at any time $t \in (0, t^*)$. Besides, by the monotonity of $L(t)$ we have that $L\beta = 1(t) \geq L\beta \geq \underline{L}$, which implies that the minimum value of the trajectory $L_{\beta=1}(t))$ is greater than or equal to $\underline{\text{L}}$, consequently $\min_{t \in (0,t^*)} L_{\beta=1}(t) \geq L^+(L_0, S_0)\ \forall t \in (0, t^*)$.

$(\geq)$

Let be $\underline{L} = \min_{t \in (0,t^*)} L_{\beta=1}(t)$, then there is $\beta = 1$ such that $L_\beta \geq \underline{L}$ and hence by definition, $\underline{L} \in A$, then $\underline{L} \leq \max A$, thus $\underline{L} = \min_{t \in (0,t^*)} L_{\beta=1}(t) \leq L^+(L_0, S_0)$.

□

Now, if we consider the restriction for the level of people, by Theorem 3.2, we have that if the dynamics of the system are run with the maximum control, i.e.,

$\beta = 1$, then $L_1(t) \geq \underline{L}$ $\forall t \in (0, t^*)$ with $0 \leq \underline{L} \leq L^+(L_0, S_0)$, which implies that in the same interval all trajectories $S(t)$ with any other control will be above of the minimum value of the trajectory of S; thus, the maximum level of resource reserve that can be sustained is defined as indicated in Proposition 3.6.

Proposition 3.6 *Let (L_0, S_0) be an initial condition, if $0 \leq \underline{L} \leq L^+(L_0, S_0)$, then*

$$S^+(L_0, S_0, \underline{L}) = \min_{t \in (0,t^*)} S_{\beta=1}(t).$$

Proof Let be $a = \min_{t \in (0,t^*)} S_{\beta=1}(t)$. Suppose that $L(t) \geq \underline{L}$ at any time $t \in (0, t^*)$ with $0 \leq \underline{L} \leq L^+(L_0, S_0)\}$. Let us see that $S^+(L_0, S_0, \underline{L}) = a$.

($\leq$)

Suppose $S^+(L_0, S_0, \underline{L})$ is a sustainable threshold, so there is a control $0 \leq \beta \leq 1$ such that the associated trajectory (L_β, S_β) starting at (L_0, S_0) satisfies the system dynamics and the constraints $L_\beta \geq \underline{L}$ and $S_\beta \geq S^+(L_0, S_0, \underline{L})$ at any time $t \in (0, t^*)$. If $\beta = 1$, then $S_1(t) \geq S^+(L_0, S_0, \underline{L})$; particularly, the minimum observed value of the trajectory of $S_1(t)$ meets this restriction in the same interval, thus $S^+(L_0, S_0, \underline{L}) \leq a$.

($\geq$)

If $0 \leq \beta \leq 1$, consider a control $\beta_1 = 1$ and a $\beta_2 < 1$, then by Proposition 3.1, there is an interval $(0, t^*)$ where two trajectories beginning from the same starting point remain separate and, by Theorem 3.1, the trajectory associated with the greatest control ($\beta = 1$) is less than any other, especially the smallest value of $S(t)$ associated with $\beta_1 = 1$, that is, $\min S_1(t) = a \leq S_{\beta_2}(t)$ in this interval, and hence, it can be said that a is a sustainable threshold, since there is a control, in this case β_2 such that the associated trajectory is above a, and thus, a is less than or equal to the maximum of the set of sustainable thresholds that is $S^+(L_0, S_0, \underline{L})$ (Definition 3.3), consequently $S^+(L_0, S_0, \underline{L}) \geq a$.

□

Proposition 3.7 *If $\underline{L} > L^+(L_0, S_0)$, then $S^+(L_0, S_0, \underline{L}) = 0$.*

Proof Let $\underline{L}$ be a value above $L^+(L_0, S_0)$, then $\underline{L}$ is not a sustainable threshold, since by definition, $L^+(L_0, S_0)$ (Definition 3.4) is the maximum of the sustainable thresholds, which implies that for whatever control, there is an instant $t \in (0, t^*)$ at which $L(t) \leq \underline{L}$, and thus there is no such a control that leads to satisfy the restriction for $L(t)$ and $S(t)$ at the time, but by definition, it is considered that always $S(t) \geq 0$ and then $S^+(L_0, S_0) = 0$. □

3.3 Viability: Equilibrium Points

The equilibrium points of a system are viable points [27]. In Sect. (2.3), maximum two equilibria were found where the human population and the natural resource for the system (14) coexist, and these equilibria are viable as long as they satisfy the viability restrictions (15); thus, the following must be fulfilled:

$$L_i(t) \geq L^+(L_0, S_0) \text{ y } S_i(t) \geq S^+(L_0, S_0, \underline{L}). \tag{16}$$

4 Results

Simulations were performed for parameter values in Table 1; the values are taken from [16]. An initial condition was considered such that the dynamic of L is increasing with respect to the control β.

In Fig. 2, the brown area bounded by $L^+(L_0, S_0)$ and $S^+(L_0, S_0, \underline{L})$ represents the set of thresholds that are sustainable. That is, given an initial condition (L_0, S_0), there is a control β, which makes the trajectory that starts in this condition satisfy the dynamics of the system and the restrictions that these thresholds involve at any time $t \in (0, t^*)$. Specifically, Fig. 2a, b shows the sets of thresholds $\underline{L}$ and $\underline{S}$, considering as initial condition the carrying capacity $S_0 = 12000$ and the initial value of the population $L_0 = 500$ y $L_0 = 1500$, respectively. It is worth noting that, the initial condition for L corresponds to the maximum threshold $L^+(L_0, S_0)$; this is because the trajectory of L with the maximum control is increasing in the interval $(0, t^*)$.

The points that are outside U_{t^*} are thresholds that regardless of the control used, at some point, some of the restrictions given by these thresholds are not satisfied. That is, the trajectory of L is not above $\underline{L}$ and/or the trajectory of S is not above $\underline{S}$.

As a further result, the U_{t^*} was found considering $\alpha_2 = 0.0003$, which means evaluating how an increase in the speed of extraction affects the set of sustainable thresholds. The results are shown in Figs. 2c, d. When comparing these results with those shown in Figs. 2a, b, a decrease of 60.6% in the values of $S^+(L_0, S_0, \underline{L})$ is observed; this is because a higher extraction rate reduces the level of resources available more quickly.

The time t^* during which U_{t^*} is defined is relevant since from that moment the monotonicity of the trajectories of the state variables is not fulfilled. In Fig. 3, it can be seen that this time regardless of the initial conditions is shorter for $L(t)$, which means that the monotonicity of $L(t)$ implies the monotonicity of $S(t)$, which is the result of Theorem 3.2.

Table 1 Parameter values

Parameter	Value	Parameter	Value
		k_2	700
ϕ	3	ρ	0.025
α_1	12.95	γ	0.1
α_2	0.0001	δ	0.7
k_1	12000	σ	1.4

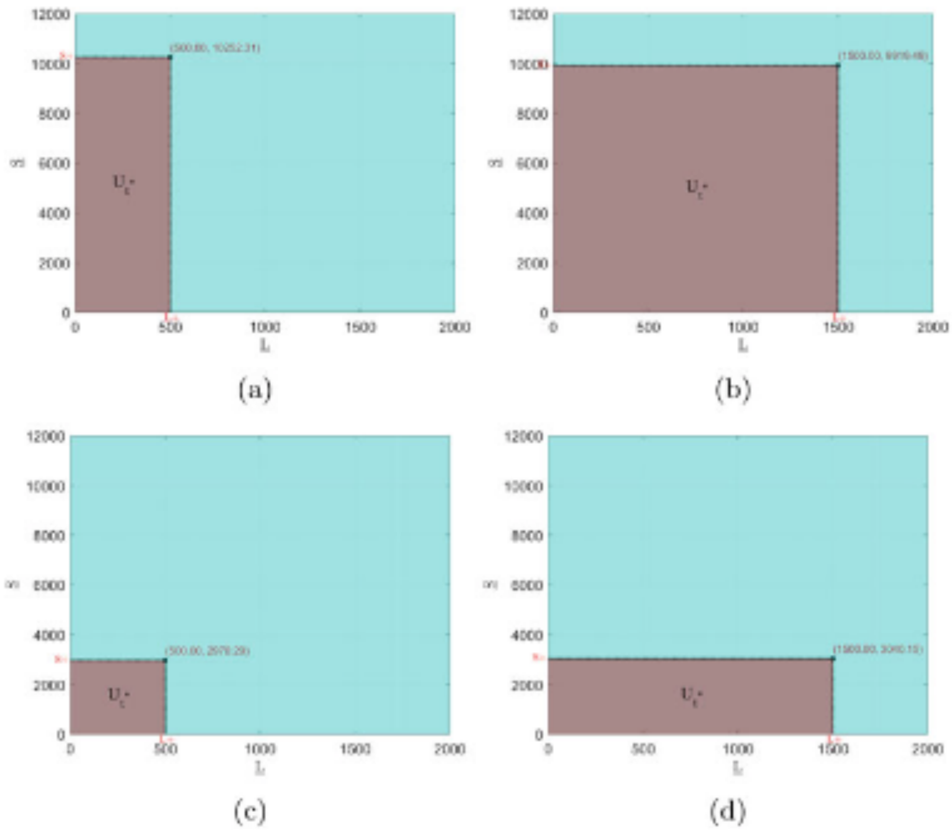


Fig. 2 Sustainable thresholds for different initial conditions in L and $\alpha_2 = 0.0003$

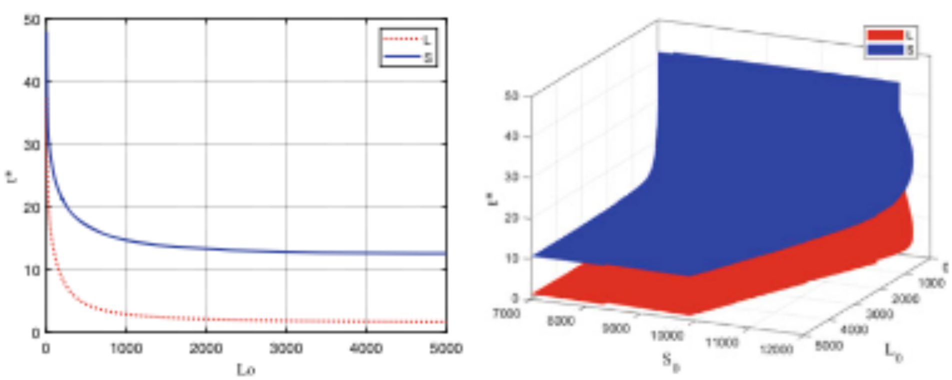


Fig. 3 Instant of time for the first crossing of the L and S trajectories with different initial conditions

With the previous results, we can say that U_{t^*} is sensitive to changes, not only in the values of the system parameters but also to changes in the initial conditions. In Fig. 4, the curve with coordinates $(L_i^+(L_0, S_0), S_i^+(L_0, S_0, \underline{L}))$ for $L_0 \in (0, 5000]$

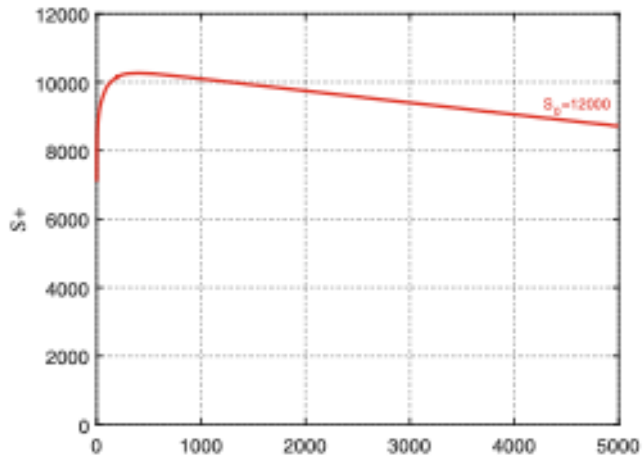


Fig. 4 Maximum sustainable thresholds for different initial conditions L_0

and $S_0 = 12000$ is shown. Taking into account that $L^+(L_0, S_0) = L_0$ because $L(t)$ is increasing with the maximum control, it is observed that the curve has a maximum, which implies that there is an L_0 for which a maximum value of $S^+(L_0, S_0\underline{L})$ is obtained.

5 Discussion: Key Challenges and Ways Forward

From model (3), the internal equilibrium points were found analytically. Additionally, the viability theory was applied to find the sustainable thresholds U_{t^*} that define the conditions of coexistence between humans and renewable resources within a time interval $(0, t^*)$ given an initial state. If the initial condition is an equilibrium point of the system and the trajectory originating there complies with the viability restrictions, it is said that this point is viable.

In literature in general [28, 30–34], the systems have monotonicity at every instant of time, that is, the trajectories do not cross and it is not necessary to calculate t^*. Furthermore, in these papers, it was possible to find a specific expression for sustainable thresholds. The proposal made in this chapter is focused on those models that have monotonic behavior only in the time interval $(0, t^*)$, since there is a crossing of trajectories for different controls. Specifically, we work with a model that represents the pressure that population makes on renewable resources, considering that they are extracted at a certain rate that depends on the labor force (β) involved in the activity. The extraction of these resources allows an increase in the level of population and a decrease in the level of the forestry resource, that is, the dynamics of the population is increasing and that of the resource is decreasing. Additionally, the intensity of the labor force (high β values) enables faster increases for the population and faster decreases for the resource. This means that with maximum control, the maximum trajectory for L and the minimum for S are obtained.

Analytically, the existence of the interval $(0, t^*)$ was demonstrated, in which the monotonicity of L implies the monotonicity of S; this means that the first crossing of trajectories associated with two different controls occurs in L, that is, it is guaranteed that for the same two controls, there is no crossing of the trajectories of S in the same interval.

Numerically, the value of t^* was found showing the existence of the interval and the monotonicity of the trajectories for different configurations in the initial conditions and values of α_2, which is a measure of the efficiency in which the resources are extracted. It was found that t^* is the time in which the trajectory with the maximum control intersects with another, determining the maximum sustainable thresholds $S^+(L_0, S_0, \underline{L})$ and $L^+(L_0, S_0)$ that comply with the restrictions of the system.

In this study, the value of t^* was not considered in the analysis, as it focused mainly on determining the value of sustainable thresholds. However, from practice, it is important to take into account the time during which compliance with the restrictions can be guaranteed, because large sustainable thresholds for a very short time may not be the best option for a decision-maker, but thresholds of moderate size for a long time may be. As a future paper, a viability analysis that integrates the sustainable thresholds U_{t^*} with the value of t^* is proposed.

For future papers, it is proposed to carry out a sensitivity analysis, varying not only the initial conditions but also the parameters of the system other than the β control, which allow to identify the different scenarios given a certain configuration.

It is also proposed to work on the problem for the case in which the dynamics of L is decreasing with respect to the control. This implies rethinking the problem and proposing the theory to define the corresponding sustainable thresholds.

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